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A Rational Chebyshev Functions Approach for Fredholm-Volterra Integro-Differential Equations

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Abstract The purpose of this study is to present an approximate numerical method for solving high order linear Fredholm-Volterra integro-differential equations in terms of rational Chebyshev functions under the mixed conditions. The method is based on the approximation by the truncated rational Chebyshev series. Finally, the effectiveness of the method is illustrated in several numerical examples. The proposed method is numerically compared with others existing methods where it maintains better accuracy.

Keywords. Rational Chebyshev functions, Fredholm-Volterra integro-differential equations, collocation method.

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1. Introduction

Many mathematical formulations of physical phenomena contain Fredholm and Volterra integro-differential equations (FVIDE). These equations arise in fluid dynamics, biological models, chemical kinetics and etc. Finding the exact solution of FVIDE is generally difficult, even impossible. Therefore, it is needed to obtain approximate solutions. Several numerical methods have been used such as the successive approximation method for FVIDE P. J. Kauthen [7] in 1989 introduced continuous time collocation methods for Volterra–Fredholm integral equations. M. T. Rashed [25] in 2004 used Lagrange interpolation to compute the numerical solutions of differential, integral and integro-differential equations which includes FVIDE. A. Arikoglu and I. Ozkol [2] in 2005 presented the solution of boundary value problems for integro-differential equations by using differential transform method. A. M. Wazwaz [26,27]

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in 2002 suggested a reliable treatment for mixed Volterra–Fredholm integral equations and in 2006 he presented a comparison between the modified decomposition method and the traditional methods. K. Maleknejad and Y. Mahmoudi [10] in 2003 using Taylor Polynomial solution of high-order nonlinear Volterra–Fredholm integro-differential equations.

S. Nas, S. Yalçınbas and M. Sezer [11] in 2000 applied Taylor polynomial approach for solving high-order linear Fredholm integro-differential equations. E. Boabolian, Z. Masouri and S. Hatamazadeh-Varmazyar [3] in 2008 constructed new direct method to solve non-linear Volterra-Fredholm integral and integro-differential equation using operational matrix block-pulse functions. A. ALJubory [1] in 2010 introduced some approximation method for solving Volterra-Fredholm integral and integrodifferential equation. M. Dadkah, Kajanj. M. Tavassoli and S. Mahdavi [5] in 2010 used numerical solution of nonlinear Volterra-Fredholm integro-differential equations using Legendre wavelets. R. Mohesn and S. H. Kiasoltani [14] in 2011 studied the solution of non-linear system of Volterra-Fredholm integro-differential equation by using discrete collocation method. Gherjalar H. D. and M. Hossein [6] in 2012 solved integral and integro-differential equation by using B-splines function.

On the other hand, in recent years, many authors studied the application of rational Chebyshev functions for solving different problems of differential equations and some other physical problems with variable coefficients. Approximate solutions of high-order ordinary, system of ordinary and intgro-differential equations have been presented in many papers, see for example Ramadan et al. [15-19]. Parand and Razzaghi [12] applied rational Chebyshev tau method for solving Volterra population model. Parand et al. [13] used rational Chebyshev Tau method for solving natural convection of Darcian fluid about a vertical full cone embedded in porous media with a prescribed wall temperature.

2. Definition of the problem

Let us consider the high order linear FVIDE as follows,

$$\sum_{k=0}^{m} P_k(x)y^{(k)}(x) = g(x) + \lambda_1 \int_0^a K_f(x,t)y(t)dt + \lambda_2 \int_0^x K_v(x,t)y(t)dt. \quad (2.1)$$

Under the mixed conditions

$$\begin{array}{c} \sum_{k=0}^{m-1} \sum_{j=0}^{J} d_{ij}^{k} y^{(k)}\left(b_j\right) = \alpha_i, 0 \le b_j \le a < \infty, \\ i = 0,\ 1,\ ...,\ m-1,\ j = 0,\ 1,\ ...,\ J, \end{array} \quad (2.2)$$

where $y(x)$ is an unknown function, $P_k(x)$, $g(x)$, $K_f(x,t)$ and $K_v(x,t)$ are defined on an interval $0 \le x,\, t \le a < \infty$, also d_{ij}^k, b_j and α_i are appropriate constants.
We will find the approximate solution of (2.1) by truncated RC series such that

$$y_N(x) = \sum_{n=0}^{N} a_n R_n(x), \quad (2.3)$$

where $R_n(x) = \cos n\theta$, $x = \cot^2(\theta/2)$, $\theta = 2arc\cot([x]^{1/2})$, $0 \le x < \infty$

and a_n, are rational Chebyshev coefficients to be determined and N is chosen any positive integer such that $N \geq m$.

If we use the expression $v(\mathrm{x})=\frac{\mathrm{x}-1}{\mathrm{x}+1}$ in the definition of rational Chebyshev functions, we have

$$R(x) = V(x)C^T, \tag{2.4}$$

where $R(x)$ and $V(x)$ are vectors of the form:

$$R(x) = \begin{bmatrix} R_0(x) & R_1(x) & \ldots & R_N(x) \end{bmatrix},$$

$$V(x) = \begin{bmatrix} v^0(x) & v^1(x) & \ldots & v^N(x) \end{bmatrix}.$$

Consequently, thej^{th} derivative of the matrix $R(x)$, can be obtained as

$$R^{(j)}(x) = V^{(j)}(x)C^T. \tag{2.5}$$

For more details about rational Chebyshev functions and their derivatives see [15]

3. The fundamental relations

Let us write Eq.(2.1) in the form

$$D(x) = g(x) + \lambda_1 I_f(x) + \lambda_2 I_v(x), \tag{3.1}$$

where the differential term is

$$D(x) = \sum_{k=0}^{m} P_k(x)y^{(k)}(x). \tag{3.2}$$

Fredholm integral term is

$$I_f(x) = \int_0^a K_f(x,t)y(t)dt, \tag{3.3}$$

and, Volterra integral term is

$$I_v(x) = \int_0^x K_v(x,t)y(t)dt. \tag{3.4}$$

We convert these terms and the mixed conditions (2.2) to the matrix forms in the following subsections.

3.1. Matrix relation for the differential term.

We first consider the approximate solution $y_N(x)$ of Eq. (2.1) defined by the truncated rational Chebyshev series (2.3). Then we can put expression (2.3) in the matrix form

$$[y_N(x)] = R(x)A, \tag{3.5}$$

and

$$[y_N^{(j)}(x)] = R^{(j)}(x)A, j = 0, 1, \ldots, m \leq N \tag{3.6}$$

where

$$A = \begin{bmatrix} a_0 & a_1 & \cdots & a_N \end{bmatrix}^T,$$

substituting the relation (2.5) into expression (3.6), we have the scheme of the (k)th-order derivative of the solution function $y(x)$ of the high order differential equations as

$$\left[y_N^{(k)}(x)\right] = V^{(k)}(x)C^T A\,. \tag{3.7}$$

By substituting the Eq. (3.7) into Eq. (3.2), we obtain the matrix representation of the differential term such that

$$D(x) = \sum_{k=0}^{m} P_k(x)V^{(k)}(x)C^T A\,. \tag{3.8}$$

3.2. **Matrix relation for the Fredholm integral term.** We have $K(x,t)$ as a function of two variables x,t. It can be expressed by expansion of truncated RC functions as

$$K(x,t) = \sum_{l=0}^{N}\sum_{s=0}^{N} k_{ls}R_{l,s}(x,t).$$

Based on Basu [19], we can introduce double RC functions in the following form

$$R_{l,s}(x,t) = R_l(x)R_s(t),$$

where R_l, R_s are RC functions. Therefore, the kernel function $K(x,\ t)$ can be expanded to univariate rational Chebyshev series with respect to t as follows

$$K(x,t) = \sum_{l=0}^{N}\sum_{s=0}^{N} k_{ls}R_l(x)R_s(t),$$

where

$$k_{ls} = \frac{4}{c_l c_s \pi^2}\int_0^\infty\int_0^\infty R_l(x)R_s(t)K(x,t)w(x)w(t)dxdt, l\,,\ s = 0,1,...,N.$$

Then, the matrix equation of the kernel functions $K_f(x,t)$ becomes

$$[K_f(x,t)] = R(x)K_f R^T(t), \tag{3.9}$$

where

$$K_f = \begin{bmatrix} k_{00} & k_{01} & \dots & k_{0N} \\ k_{10} & k_{11} & \dots & k_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ k_{N0} & k_{N1} & \dots & k_{NN} \end{bmatrix}.$$

The matrix representation of $y(t)$ can be given by

$$y(t) = R(t)A. \tag{3.10}$$

Substituting Eqs. (3.9), (3.10) into Eq. (3.3), we get

$$[I_f(x)] = \int_0^a R(x)K_f R^T(t)R(t)Adt = V(x)C^T K_f M_f A, \tag{3.11}$$

where

$$M_f = \int_0^a R^T(t)R(t)dt.$$

3.3. **Matrix relation for the Volterra integral part.** Similarly, let us assume that the kernel functions $K_v(x,t)$ can be expanded to univariate rational Chebyshev series with respect to t. Then the matrix form is obtained

$$[K_v(x,t)] = R(x)K_v R^T(t). \tag{3.12}$$

Substituting Eqs. (3.10), (3.12) into Eq. (3.4), we get

$$[I_v(x)] = \int_0^x R(x)K_v R^T(t)R(t)Adt = V(x)C^T K_v M_v(x)A. \tag{3.13}$$

4. The fundamental matrix equations based on collocation points

Let us define the collocation points x_r as

$$x_r = \frac{c}{N}r; \qquad r = 0, 1, \dots, N, \tag{4.1}$$

so that $0 \leq x_r \leq c < \infty; c \in IR^+$. Substituting the collocation points (4.1) into (3.7), we obtain the fundamental matrix equations for the differential term as follows

$$D(x_r) = \sum_{k=0}^{m} P_k(x_r)V^{(k)}(x_r)\, C^T A,$$

or shortly

$$\mathbf{D} = \sum_{k=0}^{m} \mathbf{P}_k \mathbf{V}^{(k)}\, C^T A, \tag{4.2}$$

where

$$\mathbf{P}_k = \begin{bmatrix} P_k(x_0) & 0 & \dots & 0 \\ 0 & P_k(x_1) & \dots & 0 \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & \dots & P_k(x_N) \end{bmatrix},$$

$$\mathbf{V}^{(k)} = \begin{bmatrix} V^{(i)}(t_0) \\ V^{(i)}(t_1) \\ \vdots \\ V^{(i)}(t_N) \end{bmatrix} = \begin{bmatrix} v^{(0)}(x_0) & v^{(1)}(x_0) & \dots & v^{(N)}(x_0) \\ v^{(0)}(x_1) & v^{(1)}(x_1) & \dots & v^{(N)}(x_1) \\ \vdots & \vdots & \ddots & \vdots \\ v^{(0)}(x_N) & v^{(1)}(x_N) & \dots & v^{(N)}(x_N) \end{bmatrix}.$$

Similarly, substituting the collocation points (4.1) into (3.11), we have

$$[I_f(x_s)] = \int_0^a R(x_s)K_f R^T(t)R(t)Adt = V(x_s)C^T K_f M_f A,$$

or shortly

$$[I_f] = \mathbf{V}C^T K_f M_f A. \tag{4.3}$$

Finally, substituting the collocation points (4.1) into (3.13), we have

$$[I_v] = V(x_s)C^T K_v M_v(x_s)A.$$

The fundamental matrix equation is obtained for Volterra integral term such that

$$[I_v] = \bar{\mathbf{V}}\mathbf{C}^T\mathbf{K}_v\mathbf{M}_v A, \tag{4.4}$$

where

$$\bar{\mathbf{V}} = \begin{bmatrix} V(x_0) & 0 & \cdots & 0 \\ 0 & V(x_1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & V(x_N) \end{bmatrix}_{(N+1\times(N+1)^2},$$

$$\mathbf{C}^T = \begin{bmatrix} C^T & 0 & \cdots & 0 \\ 0 & C^T & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & C^T \end{bmatrix}_{(N+1)^2\times(N+1)^2},$$

$$\mathbf{K}_v = \begin{bmatrix} K_v & 0 & \cdots & 0 \\ 0 & K_v & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & K_v \end{bmatrix}_{(N+1)^2\times(N+1)^2},$$

$$K_v = \begin{bmatrix} k_{00} & k_{01} & \dots & k_{0N} \\ k_{10} & k_{11} & \dots & k_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ k_{N0} & k_{N1} & \dots & k_{NN} \end{bmatrix}, \quad \mathbf{M}_v = \begin{bmatrix} M_v(x_0) \\ M_v(x_1) \\ \vdots \\ M_v(x_N) \end{bmatrix}_{(N+1)^2\times(N+1)}.$$

Substituting Eqs. (4.2), (4.3) and (4.4) into Eq. (3.1), we get the fundamental matrix of equation (2.1) can be obtained as:

$$\left\{\sum_{k=0}^{m}\mathbf{P}_k\mathbf{V}^{(k)}\,C^T - \lambda_1\mathbf{V}C^T K_f M_f - \lambda_2\bar{\mathbf{V}}\mathbf{C}^T\mathbf{K}_v\mathbf{M}_v\right\}A = G. \tag{4.5}$$

Similarly, the corresponding matrix form for the conditions (2.2) is obtained, by means of the Eq. (3.7) as follows

$$\sum_{k=0}^{m-1}\sum_{j=0}^{J} d_{ij}^{k}\,V^{(k)}(b_j)C^T A = \alpha_i\ . \tag{4.6}$$

5. Method of the solution

To obtain the approximate solution of Eq. (2.1) with the mixed conditions (2.2) using the presented method, we obtain the fundamental matrix equation from equation (4.5) as follows

$$\left\{\sum_{k=0}^{m}\mathbf{P}_k\mathbf{V}^{(k)}\,C^T - \lambda_1\mathbf{V}C^T K_f M_f - \lambda_2\bar{\mathbf{V}}\mathbf{C}^T\mathbf{K}_v\mathbf{M}_v\right\}A = G.$$

Denoting the expression in parenthesis of Eq. (4.5) by W, the fundamental matrix equation for Eq. (2.1) is reduced to

$$WA = G \quad or \quad [W;G], \tag{5.1}$$

which corresponds to a system of $(N\ +1)$ linear algebraic equations with unknown rational Chebyshev coefficients $a_0, a_1, ..., a_N$. We can obtain the matrix form for the mixed conditions (2.2), by means of Eq. (4.6), briefly, as

$$UA = \alpha_i; \qquad \text{or} \qquad [U;\alpha_i] \qquad i = 0, 1, ..., m-1 \ . \tag{5.2}$$

Now, the solution of Eq. (2.1) under the conditions (2.2), can be obtained by replacing the rows of matrices (5.2) by the last m rows of the matrix (5.1). Then, we get the required augmented matrix.
If rank $\mathbf{W} = rank[\mathbf{W};\mathbf{G}] = N+1$, then we can write the matrix equation (5.1) as:

$$A = (\mathbf{W})^{-1}\mathbf{G}$$

and therefore the coefficients a_n ; n =0, 1,. . . , N are uniquely determined by Eq.(5.1).

6. Illustrative Examples

In this section, numerical examples are given to illustrate the applicability, accuracy and effectiveness of the proposed technique. All examples are performed on the computer using a program written in MATHEMATICA 7.0. The obtained numerical results are presented in the given Tables. The absolute errors, in tables, are given by the values of $|y(t) - y_N(t)|$ evaluated at selected points.
Example 1.
Let us consider the linear Volterra–Fredholm integro-differential equation

$$y^{(4)}(x) - \frac{12}{(1+x)^2}y''(x) = \frac{120 - 22\ln[11]}{11+11x} + \frac{x - \ln[1+x]}{1+x}$$

$$-\int_0^{10} \frac{t}{xt+x+t+1}y(t)dt - \int_0^{x} \frac{1}{1+x}y(t)dt, \quad 0 \le x \le 10,$$

with $y(0) = 0,\ y'(0) = 1,\ y(1) = \frac{1}{2}, y(10) = \frac{10}{11}$.
For N= 4, the collocation points are

$$x_0 = 0,\ x_1 = 2.5,\ x_2 = 5,\ x_3 = 7.5,\ x_4 = 10.$$

The fundamental matrix equation of this problem is $(\mathbf{P}_0\mathbf{V}^{(0)}C^T + \mathbf{P}_1\mathbf{V}^{(1)}C^T + \mathbf{P}_2\mathbf{V}^{(2)}C^T + \mathbf{P}_3\mathbf{V}^{(3)}C^T + \mathbf{P}_4\mathbf{V}^{(4)}C^T - \lambda_1\mathbf{V}C^TK_fM_f - \lambda_2\mathbf{V}\mathbf{C}^T\mathbf{K}_v\mathbf{M}_v)A = G$. Since $\mathbf{P}_0,\ \mathbf{P}_1,\ \mathbf{P}_3$, are zero matrices and $\mathbf{P}_4$, identity matrix then we can write the fundamental matrix equation as

$$(\mathbf{P}_2\mathbf{V}^{(2)}C^T + \mathbf{V}^{(4)}C^T - \lambda_1\mathbf{V}C^TK_fM_f - \lambda_2\bar{\mathbf{V}}\mathbf{C}^T\mathbf{K}_v\mathbf{M}_v)A = G,$$

where $\mathbf{P}_2,\ \mathbf{V},\ \mathbf{V}^{(2)},\ \mathbf{V}^{(4)}, C^T,\ K_f,\ M_f, \bar{\mathbf{V}},\ \mathbf{C}^T, \mathbf{K}_V, \mathbf{M}_v$ are matrices of order 5×5 given, for this example,

$$\mathbf{p}_2 = \begin{bmatrix} -12 & 0 & 0 & 0 & 0 \\ 0 & -\frac{48}{49} & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & -\frac{48}{289} & 0 \\ 0 & 0 & 0 & 0 & -\frac{12}{121} \end{bmatrix}, \mathbf{V} = \begin{bmatrix} 1 & -1 & 1 & -1 & 1 \\ 1 & \frac{3}{7} & \frac{9}{49} & \frac{27}{343} & \frac{81}{2401} \\ 1 & \frac{2}{3} & \frac{4}{9} & \frac{8}{27} & \frac{16}{81} \\ 1 & \frac{13}{17} & \frac{169}{289} & \frac{2197}{4913} & \frac{28561}{83521} \\ 1 & \frac{9}{11} & \frac{81}{121} & \frac{729}{1331} & \frac{6561}{14641} \end{bmatrix},$$

$$\mathbf{V}^{(2)} = \begin{bmatrix} 0 & -4 & 16 & -36 & 64 \\ 0 & -\frac{32}{343} & -\frac{64}{2401} & \frac{288}{16807} & \frac{3456}{117649} \\ 0 & -\frac{1}{54} & -\frac{1}{54} & -\frac{1}{81} & -\frac{4}{729} \\ 0 & -\frac{32}{4913} & -\frac{704}{83521} & -\frac{11232}{1419857} & -\frac{151424}{24137569} \\ 0 & -\frac{4}{1331} & -\frac{64}{14641} & -\frac{756}{161051} & -\frac{7776}{1771561} \end{bmatrix},$$

$$\mathbf{V}^{(4)} = \begin{bmatrix} 0 & -48 & 384 & -1584 & 4608 \\ 0 & -\frac{1536}{16807} & \frac{9216}{117649} & \frac{50688}{823543} & \frac{6144}{823543} \\ 0 & -\frac{1}{162} & -\frac{1}{486} & \frac{1}{486} & \frac{17}{4374} \\ 0 & -\frac{1536}{1419857} & -\frac{21504}{24137569} & -\frac{133632}{410338673} & \frac{1456128}{6975757441} \\ 0 & -\frac{48}{161051} & -\frac{576}{1771561} & -\frac{4464}{19487171} & -\frac{20352}{214358881} \end{bmatrix},$$

$$M_f = \begin{bmatrix} 10 & 5.20421 & -1.91043 & -5.39352 & -5.58564 \\ 5.20421 & 4.04478 & -0.0946552 & -3.74804 & -4.33449 \\ -1.91043 & -0.0946552 & 2.20718 & 0.964378 & -1.07064 \\ -5.39352 & 3.74804 & 0.964378 & 4.88458 & 3.75714 \\ -5.58564 & 4.33449 & -1.07064 & 3.75714 & 6.72137 \end{bmatrix},$$

$$K_f = \begin{bmatrix} 0.25 & 0.25 & 0 & 0 & 0 \\ -0.25 & -0.25 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad C^T = \begin{bmatrix} 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & -3 & 0 \\ 0 & 0 & 2 & 0 & -8 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 8 \end{bmatrix}$$

$$\bar{\mathbf{V}} = \begin{bmatrix} V(x_0) & 0 & 0 & 0 & 0 \\ 0 & V(x_1) & 0 & 0 & 0 \\ 0 & 0 & V(x_2) & 0 & 0 \\ 0 & 0 & 0 & V(x_3) & 0 \\ 0 & 0 & 0 & 0 & V(x_4) \end{bmatrix}, \mathbf{C}^T = \begin{bmatrix} C^T & 0 & 0 & 0 & 0 \\ 0 & C^T & 0 & 0 & 0 \\ 0 & 0 & C^T & 0 & 0 \\ 0 & 0 & 0 & C^T & 0 \\ 0 & 0 & 0 & 0 & C^T \end{bmatrix}$$

$$\mathbf{M}_v = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 2.5 & -0.00552594 & -1.80782 & -0.457897 & 0.817805 \\ -0.00552594 & 0.346091 & -0.231711 & -0.495007 & 0.200381 \\ -1.80782 & -0.231711 & 1.6589 & 0.426566 & -0.951389 \\ -0.457897 & -0.495007 & 0.426566 & 1.20252 & -0.342655 \\ 0.817805 & 0.200381 & -0.951389 & -0.342655 & 1.03773 \\ 5 & 1.41648 & -2.66741 & -2.80723 & -0.978278 \\ 1.41648 & 1.1663 & -0.695372 & -1.82284 & -0.858094 \\ -2.66741 & -0.695372 & 2.01086 & 1.25376 & -0.415235 \\ -2.80723 & -1.82284 & 1.25376 & 3.41847 & 1.31966 \\ -0.978278 & -0.858094 & -0.415235 & 1.31966 & 2.44967 \\ 7.5 & 3.21987 & -2.56171 & -4.4468 & -3.43683 \\ 3.21987 & 2.46915 & -0.613464 & -2.99927 & -2.63082 \\ -2.56171 & -0.613464 & 2.03159 & 1.20252 & -0.516946 \\ -4.4468 & -2.99927 & 1.20252 & 4.51391 & 2.93467 \\ -3.43683 & -2.63082 & -0.516946 & 2.93467 & 4.86811 \\ 10 & 5.20421 & -1.91043 & -5.39352 & -5.58564 \\ 5.20421 & 4.04478 & -0.0946552 & -3.74804 & -4.33449 \\ -1.91043 & -0.0946552 & 2.20718 & 0.964378 & -1.07064 \\ -5.39352 & -3.74804 & 0.964378 & 4.88458 & 3.75714 \\ -5.58564 & -4.33449 & -1.07064 & 3.75714 & 6.72137 \end{bmatrix}$$

$$\mathbf{K}_v = \begin{bmatrix} K_v & 0 & 0 & 0 & 0 \\ 0 & K_v & 0 & 0 & 0 \\ 0 & 0 & K_v & 0 & 0 \\ 0 & 0 & 0 & K_v & 0 \\ 0 & 0 & 0 & 0 & K_v \end{bmatrix}.$$

We obtain the rational Chebyshev coefficient of this equation in the form

$$A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \end{bmatrix}^T.$$

Therefore, we find the solution $y(x) = \sum_{n=0}^{4} a_n R_n(x)$, to be of the form $y(x) = \frac{1}{2}R_0(x) - \frac{1}{2}R_1(x)$. Therefore, we find the solution $y(x) = \frac{x}{x+1}$, which is the exact solution of Example 1.

Example 2.

Let us consider the linear Volterra–Fredholm integro-differential equation

$$y'(x) = -e^{-x} + \frac{1-e}{e(x+1)} + \cosh(x) - \sinh(x)$$

$$-1 + \int_0^1 \frac{1}{x+1} y(t)dt + \int_0^x y(t)dt,$$

with $y(0) = 1$ and the exact solution $y(x) = e^{-x}$.
If we follow steps in section 5 for $N = 5, 7$ and 10 and compare the obtained numerical results with the exact solution you can find that, as shown in Table 1, our method gives good accuracy to type such type of problems. Also, from Table 2, it is clear that as N increases the accuracy improves .

Table 1. Comparison between Exact solution and approximate solutions obtained by present method for $y(x)$ of Example 2.

x	Exact solution	at $N=5$	at $N=7$	at $N=10$
0	1	1	1	1
0.1	0.904837418	0.904735572	0.904833387	0.904837210
0.2	0.818730753	0.818577625	0.818725994	0.818730524
0.3	0.740818220	0.740656480	0.740812982	0.740817960
0.4	0.670320046	0.670145362	0.670314126	0.670319754
0.5	0.606530659	0.606332907	0.606524092	0.606530335
0.6	0.548811636	0.548590222	0.548804383	0.548811277
0.7	0.496585303	0.496344666	0.496577297	0.496584907
0.8	0.449328964	0.449068145	0.449320175	0.449328528
0.9	0.406569659	0.406273376	0.406560004	0.406569659
1	0.367879441	0.367512984	0.367868595	0.367879441

Table 2. Comparing the L_2, L_∞ errors

	L_2	L_∞
Present Method $N = 5$	4.24663e-007	0.000101846
Present Method $N = 7$	3.71995e-010	4.03059e-006
Present Method $N = 10$	8.49706e-013	2.07843e-007

Example 3. Consider Volterra integro-differential equation [9]

$$y'(x) = 1 - \int_0^x y(t)dt,$$

with $y(0) = 0$ and the exact solution $y(x) = \sin(x)$. The numerical solutions obtained for $N = 16$ is compared with that obtained by the Block Pulse Functions and Operational Matrices [9] for $m = 64$ and 128 as shown in Tables 3. From Table 4 we find that our method gives better accuracy.

Table 3. Comparison between Exact solution and approximate solutions obtained by present method and other existed method for $y(x)$ of Example 3.

x	Exact solution	Method in [9] $m = 64$	Method in [9] $m = 128$	Present Method $N = 16$
0.1	0.099833	0.101383	0.097500	0.099806
0.2	0.198669	0.194063	0.197901	0.198649
0.3	0.295520	0.299980	0.296263	0.295505
0.4	0.389418	0.387959	0.391571	0.389407
0.5	0.479426	0.486243	0.482844	0.479416
0.6	0.564642	0.565904	0.562700	0.564636
0.7	0.644218	0.640595	0.646312	0.644214
0.8	0.717356	0.720580	0.717892	0.717355
0.9	0.783327	0.782319	0.784773	0.783329

Table 4. Comparing the L_2, L_∞ errors

	L_2	L_∞
Method in [9] $m = 64$	0.00423398	0.007812
Method in [9] $m = 128$	0.00220667	0.003906
Present Method $N = 16$	9.5663×10^{-11}	0.0000271053

Example 4. Consider Fredholm integro-differential equation [9]

$$y'(x) = -e^{-x} + e^{-1} - 1 + \int_0^1 y(t)dt,$$

with $y(0) = 1$, and the exact solution $y(x) = e^{-x}$. The numerical solutions obtained for $N = 8$ is compared with the results, using Block Pulse Functions and Operational Matrices [9] for $m = 16$ and 64 in Table 5.

Table 5. Comparing the L_2, L_∞ errors

	L_2	L_∞
Method in [9] m=16	0.013792	0.030281
Method in [9] m=64	0.00350677	0.007752
Present Method $N = 8$	1.09199×10^{-16}	5.92121×10^{-8}

Table 6. Comparison between Exact solution and approximate solutions obtained by present method and other existed method for $y(x)$ of Example 4.

x	Exact solution	Method in [9] $m=16$	Method in [9] $m=64$	Present Method $N=8$
0	1	0.969719	0.992248	1
0.1	0.904837	0.910993	0.903455	0.904837
0.2	0.818731	0.804005	0.822608	0.818731
0.3	0.740818	0.755324	0.737384	0.740818
0.4	0.670320	0.666636	0.671399	0.670320
0.5	0.606530	0.588375	0.601842	0.606530
0.6	0.548812	0.552766	0.547986	0.548812
0.7	0.496585	0.487894	0.498951	0.496585
0.8	0.449329	0.458378	0.447261	0.449329
0.9	0.406570	0.404606	0.407240	0.406570

Example 5. Let us consider the linear Volterra–Fredholm integro-differential equation

$$\begin{aligned} y'(x) = {} & \tfrac{2}{(1+t)^2} - \tfrac{x - x\log[4]}{1+2x} - \tfrac{(2x-1)(x-2\log[1+x])}{1+x} \\ & + \int_0^1 \tfrac{x}{2x+1} y(t)dt + \int_0^x \tfrac{2x-1}{x+1} y(t)dt, \end{aligned}$$

with $y(0) = -1$ and the exact solution $y(x) = \frac{x-1}{x+1}$. When we compare the absolute error functions obtained by present method of this example at N=7 and 10 with the exact solution as shown in Table 7. We find that the accuracy improves proportionally as N increases as it is clear from Table 8.

Table 7. Comparison between absolute error functions obtained by present method for $y(x)$ of Example 5 for N=7 and 10

x	e_7	e_{10}
0.1	1.13581 e-008	1.28087 e-010
0.2	2.76309 e-008	1.07782 e-010
0.3	3.95459 e-008	8.15997 e-011
0.4	7.70416 e-009	3.14594 e-011
0.5	3.38876 e-008	2.26566 e-010
0.6	5.83863 e-008	2.75379 e-010
0.7	5.7505 e-008	1.63911 e-010
0.8	3.52009 e-008	1.07457 e-011
0.9	5.942 e-011	1.50656 e-010
1.0	4.01776 e-008	2.02736 e-010

Table 8. Comparing the L_2, L_∞ errors

	L_2	L_∞
Present Method $N=7$	5.10471×10^{-15}	5.8386×10^{-8}
Present Method $N=10$	1.29975×10^{-19}	2.75379×10^{-10}

7. Conclusion

In this paper the use of rational Chebyshev (RC) collocation method for solving high-order linear FVIDE with variable coefficients is investigated. The high-order linear FVIDE and the given conditions are transformed to matrix equations with unknown rational Chebyshev coefficients. A considerable advantage of the proposed technique is that the RC coefficients of the solution are found very easily by using computer programs, especially if kernel $K_f(x,t)$ and $K_v(x,t)$ are defined on an interval $0 \leq x, t \leq a < \infty$. This variant or improvement for the method gave us a faster and more accurate method compared to the other methods. In addition, an interesting feature of this method is to find the analytical solutions if the equation has an exact solution that is a rational functions. Illustrative examples are used to demonstrate the applicability and the effectiveness of the proposed technique.

Future work: the work introduced in this paper can be extended to the infinite domain by changing the basis function by the so called exponential Chebyshev functions where our research group reported some papers about this topic for ordinary, system and partial differential equations defined in the unbounded domain [8], [20-24].

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