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Existence of positive solution to a class of boundary value problems of fractional differential equations

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Abstract This paper is devoted to the study of establishing sufficient conditions for existence and uniqueness of positive solution to a class of non-linear problems of fractional differential equations. The boundary conditions involved Riemann-Liouville fractional order derivative and integral. Further, the non-linear function f contain fractional order derivative which produce extra complexity. Thank to classical fixed point theorems of nonlinear alternative of Leray-Schauder and Banach Contraction principle, sufficient conditions are developed under which the proposed problem has at least one solution. An example has been provided to illustrate the main results.

Keywords. Boundary value problem, Existence and uniqueness results, Fractional differential differential equations, Classical fixed point theorem..

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1. Introduction

Fractional differential equations have many applications in various field of science and technology. The important applications, we may observe in various scientific and engineering disciplines like, physics, chemistry, biology, photoelasticity, control theory and signal processing. Moreover, many applications are found in dynamics, aerodynamics, electrostatics and biophysics, economics, polymers rheology, thermodynamics and biomedical science, (see [9, 11, 14, 15, 19, 26, 25]). Due to the aforementioned applications, the concerned subject has got much attention from the researchers of mathematics, physics etc. The researchers has been studied by details its theory of existences, numerical analysis and qualitative analysis. Plenty of work can be found on the said area,for more detail, we refer the reader to [1, 2, 3, 13, 7, 6, 17, 24, 26]. It

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is to be noted that Caputo fractional derivatives plays a vital role in the modeling of dynamical and physical problems as it has known physical interpretation for initial and boundary value problems like classical differential equations. But on the other hand derivative in Riemann-Liouville sense also plays important role in the area of applied analysis. Existence theory for real world problems carried out by researchers of mathematics for fractional differential equations with boundary conditions has been explored and many research articles can be found regarding this topic, for detail (see [8, 4, 18, 28, 10, 20, 27, 21, 22, 23]). In [5], Bai studied existence and multiplicity of positive solutions to the following boundary value problem

$$\begin{aligned} &\mathcal{D}^{\alpha}u(t) + a(t)f(t, u'(t)) = 0; \quad 0 < t < 1, n-1 \leq \alpha < n, n > 2, \\ &u(t)|_{t=0} = u^{'}(t)|_{t=0} = u^{''}(t)|_{t=0} = u^{''}(t)|_{t=1} = 0, \end{aligned}$$

where $f : [0,1] \times [0,\infty) \to (-\infty, +\infty)$ is nonlinear continuous function and $\mathcal{D}^{\alpha}$ is the Riemann-Liouvilli fractional derivative. In [16], Kosmative studied existence and uniqueness of solutions to two-point boundary value problem for fractional differential equations of the form

$$\begin{aligned} &\mathcal{D}^{\alpha}u(t) + f(t, u(t), u^{'}(t)) = 0; \quad t \in (0,1), 1 < \alpha < 2, \\ &u(t)|_{t=0} = u(t)|_{t=1} = 0, \end{aligned}$$

where $\mathcal{D}^{q}$ is the Riemann-Liouvilli fractional derivative. In these cited papers, the nonlinearity f involved the classical order derivative. The case where the nonlinearity f explicitly depends of fractional order derivative is important theoretically as well as in application point of view and requires more efforts to study existence results. Goodrich [12], proved multiple solutions by using Schauder's fixed point theorem for fractional differential equation of the form

$$\begin{aligned} &-\mathcal{D}^{\alpha}u(t) = f(t, u(t)); \quad 0 < t < 1, n-1 < \alpha \leq n, n \in \mathbb{N}, \\ &u^{(i)}(t)|_{t=0} = \mathcal{D}^{\beta}u(t)|_{t=1} = 0, \end{aligned}$$

where $0 \leq i \leq n-2,\ 1 \leq \beta \leq n-2, \quad \alpha > 3$. The nonlinear function f : $[0,1] \times [0,\infty) \to [0,\infty)$ is continuous.

Motivated by the above mentioned work, this paper is concerned to establish conditions for existence and uniqueness of solution to fractional order differential equation with boundary condition provided by

$$\begin{aligned} &\mathcal{D}^{\alpha}u(t) = f(t, u(t), \mathcal{D}^{p}u(t)) = h(t); \quad 2 < \alpha < 3,\ 0 < p < 1, \\ &\mathcal{I}^{3-\alpha}u(t)|_{t=0} = \mathcal{D}^{\alpha-2}u(t)|_{t=0} = u(t)|_{t=1} = 0. \end{aligned} \tag{1.1}$$

The required conditions are obtained by using classical fixed point theorem like nonlinear alternative of Leray-Schauder type and Banach contraction theorem. In last, the paper is enriched by providing a suitable example to illustrate the main results.

The paper is organized: In second section, some preliminary results are provided for obtaining the main results. In third Section, we provided ,main results in which derivation of the Green's function, integral formulations of the problem, necessary and sufficient conditions for the existence and uniqueness of solutions by applying some classical fixed point theorems. Section 4 is devoted to illustrate the main results by an example.

2. Preliminaries

We recall some basic definitions and known results from fractional calculus, functional analysis, which can be found in [14, 26, 21, 28, 27].

Definition 2.1. The fractional integral of order $q \in \mathbb{R}_+$ of the function $u : (0, \infty) \to R$ is defined as

$$\mathcal{I}^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s)\, ds,$$

where the integral on the right side is pointwise defined on $(0, \infty)$.

Definition 2.2. The Riemann-Liouville fractional order derivative of a function u on the interval $[a, b]$ is defined by

$$\mathcal{D}^\alpha u(t) = \frac{1}{\Gamma(n-\alpha)} (\frac{d}{dt})^n \int_a^t (t-s)^{n-\alpha-1} u(s)\, ds,$$

where $n = [\alpha] + 1$. It is to be noted that integral on the right hand side is pointwise defined on $(0, \infty)$.

The following Lemmas are necessary in this paper.

Lemma 2.3. [17] *The fractional order differential equation of order $\alpha > 0$ of the form*

$$\mathcal{D}^\alpha u(t) = 0,\, n-1 < \alpha \leq n,$$

has a unique solution of the form

$$u(t) = C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + ... + C_n t^{\alpha-n}, \text{where } C_i \in R,$$

$i = 1, 2, ..., n,$ *and* $n = [\alpha] + 1,$ *where* $[\alpha]$ *represents integer part of* α.

Lemma 2.4. [17] *The following result holds for a fractional derivative and integral of order α*

$$I^\alpha \mathcal{D}^\alpha u(t) = u(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + ... + C_n t^{\alpha-n},$$

where $C_i \in R$, $i = 1, 2, \ldots, n$, *and* $n = [\alpha] + 1$, *where* $[\alpha]$ *represents integer part of* α.

To establish, the main results, we need the following Theorems in this paper.

Theorem 2.5. ***(The nonlinear alternative of Leray-Schauder type*** [27]***)*** *Let D be a closed convex subset of a Banach space X. Consider a relative open subset C of D such that $0 \in C$ and let $T : C \to D$ be continuous and compact mapping. Then either*

1. *the mapping T has a fixed point in C; or*
2. *there exist $u \in \partial C$ and $\lambda \in (0, 1)$ with $u = \lambda T u$.*

Theorem 2.6. ***(The Banach contraction theorem*** [27]***)*** *Let D be a closed subset of a Banach space X and $T : D \to D$. Then T has a unique fixed point u in D and for any initial value $u_0 \in D$, the successive approximation converges to u_0 if*

$$\|Tu - T\bar{u}\| \leq k\|u - \bar{u}\|, \ forall\ u, \bar{u} \in D \ with\ 0 < k < 1.$$

3. Main results

Theorem 3.1. *Let $h(t) \in C[0, 1]$, then the boundary value problem*

$$\begin{aligned} &\mathcal{D}^{\alpha}u(t) = h(t), 2 < \alpha < 3, \ t \in [0, 1], \\ &\mathcal{I}^{3-\alpha}u(t)|_{t=0} = \mathcal{D}^{\alpha-2}u(t)|_{t=0} = u(t)|_{t=1} = 0, \end{aligned} \tag{3.1}$$

has a unique solution given by

$$u(t) = \int_0^1 G(t, s)h(s)ds,$$

where $G(t, s)$is the Green's function given as

$$G(t, s) = \frac{1}{\Gamma(\alpha)} \begin{cases} (t - s)^{\alpha-1} - t^{\alpha-1}(1 - s)^{\alpha-1}, & 0 \leq s \leq t \leq 1; \\ -t^{\alpha-1}(1 - s)^{\alpha-1}, & 0 \leq t \leq s \leq 1. \end{cases}$$

.

Proof. Inview of Lemma(2.4), boundary value problem(3.1) is written as

$$u(t) = C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + C_3 t^{\alpha-3} + \mathcal{I}^{\alpha}h(t). \tag{3.2}$$

Now for boundary conditions, (3.2) implies that

$$\begin{aligned} \mathcal{I}^{3-\alpha}u(t) &= C_1\mathcal{I}^{3-\alpha}t^{\alpha-1} + C_2\mathcal{I}^{3-\alpha}t^{\alpha-2} + C_3\mathcal{I}^{3-\alpha}t^{\alpha-3} + \mathcal{I}^{3-\alpha}\mathcal{I}^{\alpha}h(t) \\ &= C_1\frac{\Gamma(\alpha - 1 + 1)}{\Gamma(3 - \alpha + \alpha - 1 + 1)}t^{3-\alpha+\alpha-1} + C_2\frac{\Gamma(\alpha - 2 + 1)}{\Gamma(3 - \alpha + \alpha - 2 + 1)}t^{3-\alpha+\alpha-2} \\ &+ C_3\frac{\Gamma(\alpha - 3 + 1)}{\Gamma(3 - \alpha + \alpha - 3 + 1)}t^{3-\alpha+\alpha-3} + \mathcal{I}^{3}h(t) \\ \mathcal{I}^{3-\alpha}u(t) &= C_1\frac{\Gamma(\alpha)}{\Gamma(3)}t^2 + C_2\frac{\Gamma(\alpha - 1)}{\Gamma(2)}t + C_3\frac{\Gamma(\alpha - 2)}{\Gamma(1)} + \mathcal{I}^{3}h(t). \end{aligned} \tag{3.3}$$

Inview of $\mathcal{I}^{3-\alpha}u(t)|_{t=0} = 0$ and also we have $\mathcal{I}^{3}h(t) \to 0$ as $t \to 0$, in(3.3), we get $C_3 = 0$. Now for boundary conditions $D^{\alpha-2}u(t)|_{t=0=0} = 0$, (3.2)implies that

$$\begin{aligned} \mathcal{D}^{\alpha-2}u(t) &= \mathcal{D}^{\alpha-2}\left[C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \mathcal{I}^{\alpha}h(t)\right] \\ &= C_1\frac{\Gamma(\alpha)}{\Gamma(\alpha - 1 - \alpha + 2 + 1)}t^{\alpha-1-\alpha+2} + C_2\frac{\Gamma(\alpha - 1)}{\Gamma(\alpha - 2 - \alpha + 2 + 1)}t^{\alpha-2-\alpha+2} + \mathcal{I}^{2}h(t) \\ &= C_1\frac{\Gamma(\alpha)}{\Gamma(2)}t + C_2\frac{\Gamma(\alpha - 1)}{\Gamma(1)} + \mathcal{I}^{2}h(t) \end{aligned} \tag{3.4}$$

which on application of $\mathcal{D}^{\alpha-2}u(t)|_{t=0} = 0$, and $\mathcal{I}^2 h(t) \to 0$ as $t \to 0$, we get $C_2 = 0$. Hence (3.2) implies that

$$u(t) = C_1 t^{\alpha-1} + \mathcal{I}^\alpha h(t). \tag{3.5}$$

Using $u(t)|_{t=1} = 0$ in (3.5), we get $C_1 = -\mathcal{I}^\alpha h(1)$. Thus (3.2) becomes

$$\begin{aligned} u(t) &= -t^{\alpha-1}\mathcal{I}^\alpha h(1) + \mathcal{I}^\alpha h(t) \\ &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)ds - \frac{t^{\alpha-1}}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}h(s)ds \\ &= \int_0^1 G(t,s)h(s)ds. \end{aligned}$$

Hence the BVP (1.1) is equivalent to the following integral equation

$$u(t) = \int_0^1 G(t,s)f(s,u(s),\mathcal{D}^p u(s))ds. \tag{3.6}$$

□

Lemma 3.2. *The Green's function $G(t,s)$ defined in Theorem 3.1 satisfies the following inequality*

$$\int_0^1 |G(t,s)|ds \le \frac{2}{\Gamma(\alpha+1)}. \tag{3.7}$$

Proof. Proof is easy, so we omit it. □

Assume that the following hold,

(A_1) For any $x, y, u, v \in R$, there exists $L > 0$ such that
$$|f(t,u,v) - f(t,x,y)| \le L(|u-x| + |v-y|);$$
(A_2) $f : I \times R^2 \to R$ is continuous;
(A_3) There exists $a_0(t) \in L^1[0,1]$ such that $|a_0(t)| \le \delta$ and $b, c \ge 0$ with
$$|f(t,u,v)| \le a_0(t) + b|u|^\rho + c|v|^\theta,$$
where $0 < \rho < \theta < 1$; or
$$|f(t,u,v)| \le b|u|^\rho + c|v|^\theta \text{ for } \rho, \theta \ge 1,\ b, c > 0.$$

Throughout this paper, we use the following space defined by

$$\begin{aligned} &X = \{u(t)|u(t) \in C^1[0,1] \text{ and } \mathcal{D}^p u(t) \in C^1[0,1], 0 < p < 1\} \\ &\|u\| = \max_{t\in[0,1]} |u(t)| + \max_{t\in[0,1]} |\mathcal{D}^p u(t)|, \end{aligned}$$

then $(X, \|\cdot\|)$ is Banach space. Let define an operator $T : X \to X$ by

$$Tu(t) = \int_0^1 G(t,s)f(s,u(s),\mathcal{D}^p u(s))ds,\ t \in [0,1]. \tag{3.8}$$

Theorem 3.3. *Under the assumptions* (A_1) *to* (A_3) *BVP (1.1) has at least one solution.*

Proof. Let us define $T : X \to X$ by

$$Tu(t) = \int_0^1 G(t,s) f(s, u(s), \mathcal{D}^p u(s)) ds.$$

Define a closed set as

$$B = \{u(t)|u \in X : \ \|u\| \leq R, \ 0 \leq t \leq 1\}, \text{where } \max\left\{(3Ab)^{\frac{1}{3}-\rho}, (3Ac)^{\frac{1}{3}-\theta}, 3k\right\} \leq R,$$

where

$$\begin{aligned} A &= \frac{2\alpha - p}{\alpha\Gamma(\alpha - p + 1)} + \frac{2}{\Gamma(\alpha+1)} \\ k &= \delta\left[\frac{2}{\Gamma(\alpha+1)} + \frac{1}{\alpha\Gamma(\alpha-p)} + \frac{1}{\Gamma(\alpha-p+1)}\right]. \end{aligned}$$

For any $u \in B$ and $T : B \to B$, we have

$$\begin{aligned} |Tu(t)| &= \left|\int_0^1 G(t,s) f(s, u(s), \mathcal{D}^p u(s)) ds\right| \\ &\leq \int_0^1 |G(t,s)| f(s, u(s), \mathcal{D}^p u(s))| ds \\ &\leq \int_0^1 |G(t,s)| \left(|a_0(s)| + b|u(s)|^\rho + c|\mathcal{D}^p u(t)|^\theta\right) ds \\ &\leq \int_0^1 |G(t,s)| a_0(s)| ds + \left(bR^\rho + cR^\theta\right) \int_0^1 |G(t,s)| ds \\ &\leq \frac{2}{\Gamma(\alpha+1)} \left[\delta + bR^\rho + cR^\theta\right]. \end{aligned} \tag{3.9}$$

Now from (3.6), we have

$$\begin{aligned} \mathcal{D}^p u(t) &= \mathcal{D}^p \left(-\mathcal{I}^\alpha h(1) t^{\alpha-1} + \mathcal{I}^\alpha h(t)\right) = -\mathcal{I}^\alpha h(1) \mathcal{D}^p t^{\alpha-1} + \mathcal{D}^p \mathcal{I}^\alpha h(t) \\ &= -\mathcal{I}^\alpha h(1) \frac{\Gamma(\alpha) t^{\alpha-1-p}}{\Gamma(\alpha-p)} + \mathcal{I}^{\alpha-\beta} h(t). \end{aligned}$$

Thus by applying operator T, we have

$$|\mathcal{D}^p Tu(t)| =$$

$$\left| -\frac{t^{\alpha-p-1}}{\Gamma(\alpha-p)} \int_0^1 (1-s)^{\alpha-1} f(s,u(s),\mathcal{D}^p u(s))ds + \frac{1}{\Gamma(\alpha-p)} \int_0^t (t-s)^{\alpha-p-1} f(s,u(s),\mathcal{D}^p u(s))ds \right|$$

$$\leq \frac{1}{\Gamma(\alpha-p)} \left\{ \left[\int_0^1 (1-s)^{\alpha-1} \Big(|a_0(s)| + bR^\rho + cR^\theta \Big) ds \right] \right.$$

$$\left. + \left[\int_0^t (t-s)^{\alpha-p-1} \Big(|a_0(s)| + bR^\rho + cR^\theta \Big) ds \right] \right\}$$

$$\leq \frac{\delta}{\Gamma(\alpha-p)} \left[\frac{1}{\alpha} + \frac{1}{\alpha-p} \right] + \frac{2\alpha-p}{\alpha\Gamma(\alpha-p+1)} \Big(bR^\rho + cR^\theta \Big).$$

Which implies that

$$|\mathcal{D}^p Tu(t)| \leq \frac{\delta}{\Gamma(\alpha-p)} \left[\frac{1}{\alpha} + \frac{1}{\alpha-p} \right] + \frac{2\alpha-p}{\alpha\Gamma(\alpha-p+1)} \Big(bR^\rho + cR^\theta \Big). \tag{3.10}$$

Adding (3.9) and (3.10), using $|a_0(s)| \leq \delta$, we obtain

$$\begin{aligned} |Tu(t)| + |\mathcal{D}^p Tu(t)| &\leq \frac{2\delta}{\Gamma(\alpha+1)} + \frac{2(bR^\rho + cR^\theta)}{\Gamma(\alpha+1)} + \frac{\delta}{\Gamma(\alpha-p)} \left[\frac{1}{\alpha} + \frac{1}{\alpha-p} \right] \\ &+ \frac{2\alpha-p}{\alpha\Gamma(\alpha-p+1)} \Big(bR^\rho + cR^\theta \Big) \\ &\leq \delta \left[\frac{2}{\Gamma(\alpha+1)} + \frac{1}{\alpha\Gamma(\alpha-p)} + \frac{1}{\Gamma(\alpha-p+1)} \right] \\ &+ \left[\frac{2\alpha-p}{\alpha\Gamma(\alpha-p+1)} + \frac{2}{\Gamma(\alpha+1)} \right] \Big(bR^\rho + cR^\theta \Big) \end{aligned}$$

which implies that $\|Tu\| \leq k + (bR^\rho + cR^\theta)A \leq \dfrac{R}{3} + \dfrac{R}{3} + \dfrac{R}{3} = R.$

Also $Tu(t)$ and $\mathcal{D}^p Tu(t)$ are continuous. Thus $T : B \to B$ is well defined and bounded. Similarly using same argument for other condition of assumption (A_3), we get

$$\|Tu\| < \frac{R}{2} + \frac{R}{2} = R.$$

Now, we are going to show that T is completely continuous operator. Let $\tau \leq t \in [0,1]$, then

$$\begin{aligned} |Tu(t) - Tu(\tau)| &= \left| \int_0^1 (G(t,s) - G(\tau,s)) f(s,u,\mathcal{D}^p u)) ds \right| \\ &\leq \int_0^1 |G(t,s) - G(\tau,s)| |f(s,u,\mathcal{D}^p u)| ds. \end{aligned} \tag{3.11}$$

Let $c, d \in (\tau, t)$ for $0 \leq s \leq t$ and by the application of Lagrange's mean value theorem, we proceed as

$$\begin{aligned} |G(t,s) - G(\tau,s)| &= \frac{1}{\Gamma(\alpha)}\left[(t-s)^{\alpha-1} - (\tau-s)^{\alpha-1} + (1-s)^{\alpha-1}(\tau^{\alpha-1} - t^{\alpha-1})\right] \\ &\leq \frac{\alpha-1}{\Gamma(\alpha)}\left[(c-s)^{\alpha-2}(t-\tau) - (1-s)^{\alpha-1}d^{\alpha-2}(t-\tau)\right]. \end{aligned}$$

Similarly there exists $e \in (t, \tau)$ for $0 \leq t \leq s$

$$(\tau^{\alpha-1} - t^{\alpha-1}) \leq (\alpha-1)e^{\alpha-2}(\tau - t)(1-s)^{\alpha-1}.$$

Since $\alpha \leq 3$, therefore using $\alpha - 1 \leq 2$, then (3.11) implies that

$$\begin{aligned} &|Tu(\tau) - Tu(t)| \\ &\leq \frac{(t-\tau)}{\Gamma(\alpha)}\int_0^1 2\{(c-s)^{\alpha-1} - d^{\alpha-2}(1-s)^{\alpha-1} - e^{\alpha-2}(1-s)^{\alpha-1}\}(a_0(s) + bR^\rho + cR^\theta)ds. \end{aligned} \tag{3.12}$$

Let

$$\mathbf{G}_{c,d,e}(s) = 2\{(c-s)^{\alpha-2} - (1-s)^{\alpha-1}d^{\alpha-2} - e^{\alpha-2}(1-s)^{\alpha-1}\}.$$

Then (3.12) becomes

$$|Tu(t) - Tu(\tau)| \leq \frac{(t-\tau)}{\Gamma(\alpha)}\int_0^1 \mathbf{G}_{c,d,e}(s)(a_0(s) + bR^\rho + cR^\theta)ds. \tag{3.13}$$

Similarly one has

$$|D^p Tu(t) - \mathcal{D}^p Tu(\tau)| \leq \frac{(t-\tau)}{\Gamma(\alpha)}\int_0^1 \mathbf{G}_{c,d,e}(s)(a_0(s) + bR^\rho + cR^\theta)ds \tag{3.14}$$

Now from(3.13) and (3.14), we have

$$\|Tu(t) - Tu(\tau)\| \leq \frac{(t-\tau)}{\Gamma(\alpha)}\left[\int_0^1 \mathbf{G}_{c,d,e}(s) + \mathbf{G}_{c,d,e}(s)\right][(a_0(s) + bR^\rho + cR^\theta)ds]. \tag{3.15}$$

Now $t \to \tau$ in (3.15) gives

$$\|Tu(t) - Tu(\tau)\| \to 0.$$

Thus T is equcontinous, as T is uniformly continuous on $[0,1]$. Thus $T(B)$ is equcontinuous set and also is bounded. Moreover, $T(B) \subseteq B$. It follows that T is completely continuous.

Thus T is at least one fixed point by using nonlinear alternative of Leray-Schauder theorem 2.5, which is the corresponding solution of (1.1). This completes the proof. $\square$

Theorem 3.4. *If $(A_1), (A_2)$ hold and $L\Omega = \left(\frac{2}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha-p+1)}\right)L < 1$. Then T has a unique fixed point.*

Proof. Let $u(t), v(t) \in C^1[0,1]$, then we have

$$|Tu(t) - Tv(t)| \leq \int_0^1 |G(t,s)||f(t,u,\mathcal{D}^p u) - f(t,v,\mathcal{D}^p v)|ds$$
$$\leq \max \int_0^1 \left(\frac{(t-s)^{\alpha-1} - t^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)} \right) L(|u-v| + |\mathcal{D}^p u - \mathcal{D}^p v|)ds.$$

Thus $\|Tu - Tv\| \leq \frac{2L}{\Gamma(\alpha+1)}\|u - v\|.$

$$\tag{3.16}$$

Similarly

$$\|\mathcal{D}^p Tu - \mathcal{D}^p Tv\| \ \leq \ \frac{L}{\Gamma(\alpha - p + 1)}\|u - v\|. \tag{3.17}$$

Thus from (3.16) and (3.17), we have

$$\|Tu - Tv\| \ \leq \ \left(\frac{2L}{\Gamma(\alpha+1)} + \frac{L}{\Gamma(\alpha-p+1)} \right) \|u - v\| = L\Omega\|u - v\|.$$

Thus by applying Theorem 2.6, T has a unique fixed point which is the unique positive solution of BVP (1.1). □

4. Example

Example 1.

$$\begin{cases} \mathcal{D}^{\frac{5}{2}} u(t) = \dfrac{t}{4} + \dfrac{\sin^{\frac{1}{2}} |u(t)|}{32} + \dfrac{\cos |\mathcal{D}^{\frac{1}{2}} u(t)|}{32}, t \in [0,1], \\ \mathcal{I}^{\frac{1}{2}} u(0) = \mathcal{D}^{\frac{1}{2}} u(0) = u(1) = 0. \end{cases} \tag{4.1}$$

From (4.1), we see that $\alpha = \frac{5}{2}$ and $p = \frac{1}{2}$, where

$$\begin{aligned} f(t,u,\mathcal{D}^p u) \quad &= \quad \frac{t}{4} + \frac{\sin^{\frac{1}{2}} |u(t)|}{32} + \frac{\cos^{\frac{1}{2}} |\mathcal{D}^{\frac{1}{2}} u(t)|}{32} \\ |f(t,u,\mathcal{D}^p u)| \quad &\leq \quad \frac{1}{4} + \frac{1}{32}|u|^{\frac{1}{2}} + \frac{1}{32}|\mathcal{D}^{\frac{1}{2}} u(t)|^{\frac{1}{2}}, \end{aligned}$$

clearly $a_0 = \frac{1}{4}$, $b = \frac{1}{32}$, $c = \frac{1}{32}$, $\theta = \rho = \frac{1}{2}, u, v \in R$, then

$$|f(t,u,\mathcal{D}^p u) - f(t,v,\mathcal{D}^p v)| \leq \frac{1}{32}\left(|u - v| + |\mathcal{D}^p u - \mathcal{D}^p v|\right), \text{ where } L = \frac{1}{32}.$$

As assumptions (A_1) to (A_3) hold, so BVP (4.1) has at least one solution. While

$$\Omega = \frac{2}{\Gamma(\frac{5}{2}+1)} + \frac{1}{\Gamma(\frac{5}{2} - \frac{1}{2} + 1)} = \frac{2}{\Gamma(\frac{7}{2})} + \frac{1}{\Gamma(3)} = \frac{2}{\frac{5}{2}.\frac{3}{2}.\frac{1}{2}\Gamma(\pi)} + \frac{1}{2} = 1.10180.$$

From which, we have $L\Omega = \frac{1.10180}{32} = 0.034431 < 1$. Hence BVP (4.1) has a unique solution.

5. Conclusion

By the use of classical fixed point theorem of nonlinear alternative of Leray-Schauder type and Banach concatenation theorem, we have developed some necessary and sufficient conditions for the existence of at least one positive solutions to a class of nonlinear fractional differential equations. Where the nonlinearity of the problem explicitly depends on the fractional order derivative which produce an extra complexity and such type of problems occur in applications in real world.

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