Threshold harvesting policy and delayed ratio-dependent functional response predator-prey model

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Abstract This paper deals with a delayed ratio-dependent functional response predator-prey model with a threshold harvesting policy. We study the equilibria of the system before and after the threshold. We show that the threshold harvesting can improve the undesirable behavior such as nonexistence of interior equilibria. The global analysis of the model as well as boundedness and permanence properties are examined too. Then we analyze the effect of time delay on the stabilization of the equilibria, i.e., we study whether time delay could change the stability of a co-existence point from an unstable mood to a stable one. The system undergoes a Hopf bifurcation when it passes a critical time delay. Finally, some numerical simulations are performed to support our analytic results.

1. Introduction

Mathematical model for predator-prey interaction is studied originally by Lotka [17] and Volterra [22] as

$$\begin{cases} \dot{x} &=& \gamma x - \alpha xy, \\ \dot{y} &=& \beta xy - \delta y, \end{cases} \qquad (1.1)$$

where x and y are the numbers of prey and predator, respectively. In this classical model the positive parameters γ, α, β, and δ stand for growth rate of prey, predation rate, conversion rate to change prey biomass into predator reproduction and death rate of predator, respectively.

More generally the predator-prey model is the following system

$$\begin{cases} \dot{x} & = & rx(1 - \frac{x}{k}) - F(x,y), \\ \dot{y} & = & \beta\, F(x,y) - \delta y. \end{cases} \tag{1.2}$$

The positive parameters r, k, β and δ represent the prey intrinsic growth rate, the environmental carrying capacity, conversion rate to change prey biomass into predator reproduction and predator's death rate, respectively. The function $F(x,y)$ describes predation and is called the *functional response*.

Traditionally, $F(x,y)$ is assumed to be a function of the prey population x, that is, $F(x,y) = F(x)$, where $F(x)$ is a Holling type (II) function [18]. It is shown that a predator-prey model with the prey-dependent functional response, may expose the so-called *paradox of enrichment* or the *biological control paradox* [10, 20, 2, 9].

The following ratio-dependent functional response predator-prey model has been suggested by Arditi and Ginzburg in [3]

$$\begin{cases} \dot{x} & = & rx(1 - \frac{x}{k}) - \frac{axy}{abx+y}, \\ \\ \dot{y} & = & y\left(-d + \frac{\eta ax}{abx+y}\right). \end{cases} \tag{1.3}$$

Here $a > 0$ and $b > 0$ are predator's attack rate and handling time, respectively.

System (1.3) exposes neither the paradox of enrichment nor the biological control paradox [4, 11, 12]. One can simplify (1.3), by rescaling

$$t \to rt, \quad x \to x/k \quad y \to y/abk.$$

Therefore the ratio-dependent functional response predator-prey model is written as

$$\begin{cases} \dot{x} & = & x(1 - x) - \frac{\alpha xy}{x+y}, \\ \\ \dot{y} & = & -\delta y + \frac{\beta xy}{x+y}, \end{cases} \tag{1.4}$$

where $\alpha = \frac{a}{r}$, $\beta = \frac{\eta}{br}$, $\delta = \frac{d}{r}$.

Moreover, in point of view of human needs such as in fishery, forestry and wildlife management, the harvesting of populations is an interesting subject. Constant, linear and quadratic harvesting have been considered so far, for example see, [16, 24, 25, 21, 19].

Another harvesting policy is the threshold harvesting. It works as follows:

When the population is above of a certain level (threshold) T, the harvesting occurs; when the population falls below that level, the harvesting stops. The policy was first studied by Collie and Spencer [6], and additional analysis has been done since then [1, 14, 13]. Classically, such harvesting function for a population model is defined as a discontinuous function

$$\phi(z) = \begin{cases} 0 & z \le T, \\ \epsilon & z > T, \end{cases} \tag{1.5}$$

where $z = x$ or $z = y$. The discontinuous function (1.5) is impractical in real world, because it would be difficult for managers to harvest immediately, at a rate ϵ, once

FIGURE 1. Graph of the continuous threshold harvesting function.
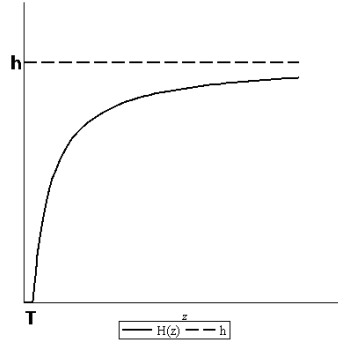
$z > T$. So the continuous threshold function proposed as the following

$$H(z) = \left\{ \begin{array}{ll} 0 & z \leq T, \\ \frac{h(z-T)}{h+z-T} & z > T, \end{array} \right. \qquad (1.6)$$

for $z = x$ or $z = y$ [15, 23]. In (1.6), T is the threshold value that determines when harvesting starts or stops and once the population passes T, then harvesting starts and increases smoothly to a limit value h, so the parameter h is the rate of harvesting limit. Because of the continuity of $H(z)$, the managers can adjust the rate of harvesting, more easily, see FIGURE 1.

On the other hand, once the predation occurs, the next generation of predator is not reproduced immediately. So the system (1.4) is impractical in real world.

In this paper we consider a ratio-dependent functional response predator-prey model with a continuous threshold harvesting and with a discrete time delay. In the model, the delay represents the time that takes to predator for consuming prey and to reproduce its next generation. We show that the feedback of the predator density (represented by time delay), might cause the oscillatory behavior.

The subject of this paper is to study the combined effects of harvesting and delay on the dynamics of a ratio-dependent predator-prey model. The reason for choosing this model is that, since we know the dynamics of the system, it will be better for us to determine the effects of delay and harvesting. Furthermore, to the best of our knowledge this is the first time that the global analysis of a delayed ratio-dependent functional response model is studied.

The paper is organized as follows. In Section 2, we determine the equilibria of the harvested and unharvested models. Some global analysis of the model, as well as boundedness and permanence is given too. In Section 3, we study the local stability of the equilibria, without time delay. In Section 4, we study the effect of the time delay in the stability of the co-existence equilibrium. We will show that the system undergoes a Hopf bifurcation when it passes a critical time delay. Some numerical

simulations have been done in Section 5, to support the analytic results. Conclusions are made in Section 6.

2. Equilibria of the ratio-dependent functional response model

In this section we consider the following delayed ratio-dependent functional response predator-prey model with a threshold harvesting policy

$$
\begin{cases}
\dot{x} & = & x(1-x) - \frac{\alpha x y}{x+y}, \\
\\
\dot{y} & = & y\left(-\delta + \frac{\beta x(t-\tau)}{x(t-\tau)+y(t-\tau)}\right) - H(y),
\end{cases}
\tag{2.1}
$$

where

$$
H(y) = \begin{cases}
0 & y \leq T, \\
\frac{h(y-T)}{h+y-T} & y > T,
\end{cases}
\tag{2.2}
$$

and the initial conditions

$$
x_0(\theta) = \phi_1(\theta) \geq 0, \ y_0(\theta) = \phi_2(\theta) \geq 0, \ \theta \in [-\tau, 0],
$$
$$
x(0) > 0, y(0) > 0,
$$

where $(\phi_1, \phi_2) \in C([\tau, 0], \mathbb{R}_+^2)$ and $x_t(\theta) = x(t+\theta)$, $y_t(\theta) = y(t+\theta)$.

In this model, the delay represents the time due to converting prey biomass into predator biomass. The function $H(y)$ is the predator threshold harvesting.

Denote N_x, N_y respectively, the prey and predator nullclines. That is

$$
N_x = \{(x,y): \ x = 0\} \cup \left\{(x,y): y = \frac{x(x-1)}{1-\alpha-x}\right\},
\tag{2.3}
$$

$$
N_y = \begin{cases}
\{(x,y): \ y = 0\} \cup \left\{(x,y): x = \frac{\delta y}{\beta-\delta}\right\} & y \leq T, \\
\\
\left\{(x,y): \ x = \frac{\delta(h+y-T)y^2+h(y-T)y}{(\beta-\delta)(h+y-T)y-h(y-T)}\right\} & y > T.
\end{cases}
\tag{2.4}
$$

As we are interested in biologically feasible equilibria, we only consider the points in $N_x \cap N_y \cap \mathbb{R}_+^2$, where $\mathbb{R}_+^2$ is the first quadrant. The system (2.1) has two boundary equilibria $O = (0,0), E = (1,0)$. Furthermore, when $y \leq T$, $N_x \cap N_y$ has another common element $E^* = (x^*, y^*)$ given by

$$
x^* = \frac{\beta-\alpha\beta+\alpha\delta}{\beta},
$$
$$
y^* = \frac{\beta-\delta}{\delta}x^* = \frac{x^*(x^*-1)}{1-\alpha-x^*} = \frac{\beta^2-\alpha\beta^2+2\alpha\delta\beta-\beta\delta-\alpha\delta^2}{\beta\delta}.
\tag{2.5}
$$

Thus the system has an interior equilibrium E^* in the first quadrant when

$$
\beta - \alpha\beta + \alpha\delta > 0, \ \beta > \delta.
\tag{2.6}
$$

Note that if $0 < \alpha < 1$, then the condition $\beta > \delta$ implies the condition $\beta-\alpha\beta+\alpha\delta > 0$.

Remark 2.1. The condition $\beta > \delta$ means that in the case $y < T$, to grant a coexistence equilibrium, the predator growth parameter β must be sufficiently larger than the predator death parameter δ.

In the next proposition, we show that for $y > T$, the system (2.1) has no prey-free equilibrium.

Proposition 2.2. *If $y > T$, then $N_y \cap \{(x, y) : x = 0\} = \emptyset$.*

Proof. Assume that $y > T$ and $(x, y) \in N_y \cap \{(x, y) : x = 0\}$. By (2.4) we have

$$\delta(h + y - T)y + h(y - T) = 0. \tag{2.7}$$

The solutions of (2.7) are

$$y = \frac{-(\delta h - \delta T + h) \pm \sqrt{(\delta h - \delta T + h)^2 + 4\delta h T}}{2\delta}, \tag{2.8}$$

and only the following solution is positive

$$y = \frac{-(\delta h - \delta T + h) + \sqrt{(\delta h - \delta T + h)^2 + 4\delta h T}}{2\delta}.$$

By the following relations it implies that $y < T$, a contradiction with the hypothesis.

$$0 > -4\delta^2 T h$$
$$\iff (\delta T + \delta h + h)^2 > (\delta h - \delta T + h)^2 + 4\delta h T$$
$$\iff \delta T + \delta h + h > \sqrt{(\delta h - \delta T + h)^2 + 4\delta h T}$$
$$\iff 2\delta T > -(\delta h - \delta T + h) + \sqrt{(\delta h - \delta T + h)^2 + 4\delta h T}$$
$$\iff T > \frac{-(\delta h - \delta T + h) + \sqrt{(\delta h - \delta T + h)^2 + 4\delta h T}}{2\delta}$$
$$\iff T > y.$$

Thus the system has no prey-free equilibrium in the first quadrant. $\qquad\square$

By Proposition 2.2, if $y > T$, the interior equilibrium of the system is the solution of the system

$$\begin{cases} y &= \frac{x(x-1)}{1-\alpha-x}, \\[2mm] x &= \frac{\delta(h+y-T)y^2 + h(y-T)y}{(\beta-\delta)(h+y-T)y - h(y-T)}. \end{cases} \tag{2.9}$$

From the expression $y = \frac{x(x-1)}{1-\alpha-x}$, we know that the points in (2.9) exist in $\mathbb{R}_+^2$, if $1 - \alpha < x < 1$. By substituting $y = \frac{x(x-1)}{1-\alpha-x}$ into the expression in N_y, we find that points in (2.9) satisfy the equation

$$\begin{aligned} G(x) :=& \beta x^5 + (\alpha\beta + \beta T - 3\beta - \beta h - \alpha\delta)x^4 + \\ &(h\alpha - 2\beta h\alpha + 2\beta\alpha T + \delta h\alpha - \delta\alpha T + 3\beta + 3\beta h - 3\beta T \\ &+ 2\alpha\delta - 2\alpha\beta)x^3 + \\ &(-\beta - \beta h\alpha^2 + 4\beta h\alpha - \alpha\delta - 3\beta h + 2\delta\alpha T - \delta\alpha^2 T + \beta\alpha^2 T \\ &+ \alpha\beta + h\alpha T + \delta h\alpha^2 + 3\beta T - 2\delta h\alpha - 4\beta\alpha T - 2h\alpha + h\alpha^2)x^2 + \\ &(2\beta\alpha T - \delta h\alpha^2 + h\alpha + \delta\alpha^2 T - 2\beta h\alpha - \beta T - \beta\alpha^2 T + \delta h\alpha \\ &+ 2h\alpha^2 T - \delta\alpha T - h\alpha^2 - 2h\alpha T + \beta h + \beta h\alpha^2)x + \\ &h\alpha T - 2h\alpha^2 T + h\alpha^3 T = 0. \end{aligned} \tag{2.10}$$

By the intermediate theorem $G(x)$ has at least one root. Denote by x^{**}, the positive solution of the Eq. (2.10) (if there exists anyone), and let

$$y^{**} = \frac{x^{**}(x^{**} - 1)}{1 - \alpha - x^{**}}. \tag{2.11}$$

If $\alpha \geq 1$ then $y^{**} = \frac{x^{**}(x^{**}-1)}{1-\alpha-x^{**}} < 1$ and we should have $T < 1$. In this case if

$$\frac{1 - T - \sqrt{(T-1)^2 - 4T(1-\alpha)}}{2} < x^{**} < \frac{1 - T + \sqrt{(T-1)^2 - 4T(1-\alpha)}}{2},$$

then $y^{**} > T$. In the case $\alpha < 1$ and $1 - \alpha - x^{**} \geq 0$, if

$$x^{**} > \frac{1 - T + \sqrt{(T-1)^2 - 4T(1-\alpha)}}{2},$$

then $y^{**} > T$. In the case $\alpha < 1$ and $1 - \alpha - x^{**} < 0$, if

$$0 < x^{**} < \frac{1 - T + \sqrt{(T-1)^2 - 4(T-T\alpha)}}{2},$$

then $y^{**} > T$. Finally if $\alpha = 1$ and $0 < x^{**} < \frac{1+\sqrt{1+4T}}{2}$, then $y^{**} > T$.

The following theorem summarized the above discussions. Note that the theorem, nevertheless, does not reveal under which conditions in harvested model an interior equilibrium appear. We rely on numerical computation to answer this question. Indeed with a numerical simulation in Section 5, we give examples at which the threshold harvesting policy can prevent the extinction of both species, prey and predator. Furthermore if $\beta > \delta$ and $\beta - \alpha\beta + \alpha\delta > 0$, then the unharvested model has an interior equilibrium (x^*, y^*). If $y^* > T$, by the last claim of the theorem the harvested model has an interior equilibrium too.

Theorem 2.3. *The boundary equilibria of the system (2.1) in the first quadrant are the co-extinction point $O = (0,0)$ and the predator-free point $E = (1,0)$. If $\beta > \delta$ and $\beta - \alpha\beta + \alpha\delta > 0$, then the unharvested model has a co-existence equilibrium $E^* = (x^*, y^*)$ defined by (2.5). Furthermore if $y^* \leq T$, then E^* is an equilibrium of the harvested model too. If $y^* > T$ and $(x^{**}, y^{**}) \in \mathbb{R}^2_+$, then the harvested model has a co-existence equilibrium $E^{**} = (x^{**}, y^{**})$ defined by (2.9) and we have $x^{**} > x^*$, $T < y^{**} < y^*$.*

Proof. By the above discussions, we only prove the last claim, which is true by the following lemmas.

Lemma 2.4. *Let $\hat{x} = \frac{\delta y}{\beta - \delta}$, $\tilde{x} = \frac{\delta(h+y-T)y^2 + h(y-T)y}{(\beta-\delta)(h+y-T)y - h(y-T)}$ for a fixed $y \geq 0$. If $y > T$, then $\tilde{x} > \hat{x}$.*

Proof. The following relations prove the result.

$$0 < hy(\beta - \delta)(y - T) \Leftrightarrow$$
$$\delta(\beta - \delta)(h + y - T)y^2 - \delta hy(y - T) <$$
$$\delta(\beta - \delta)(h + y - T)y^2 + (\beta - \delta)hy(y - T) \Leftrightarrow$$
$$\frac{\delta y}{\beta - \delta} < \frac{\delta(h+y-T)y^2 + h(y-T)y}{(\beta-\delta)(h+y-T)y - h(y-T)} \Leftrightarrow$$

$$\hat{x} < \tilde{x}.$$

$\square$

Comparing the harvested and unharvested systems, since $H(y) > 0$ in $[T, +\infty)$ one can easily prove the following lemma.

Lemma 2.5. *Let y^*, y^{**} be the second components of the positive equilibrium of the harvested and unharvested model respectively. If $y^* > T$, then $y^{**} < y^*$.*

$\square$

In other word, while the unharvested coexistence equilibrium is stable, the threshold harvesting policy can never increase both populations. In this case coexistence equilibrium have a larger prey population and a lower predator population. With a numerical simulation in Section 5, we give examples at which the threshold harvesting policy can prevent the extinction of both species, prey and predator.

In the rest of this section we study the global qualitative behavior of system (2.1).

Lemma 2.6. *The first quadrant is invariant for system (2.1).*

Proof. Suppose that there exists $A > 0$, such that for all $t \in [0, A)$, we have $x(t) > 0$, $y(t) > 0$ and either $x(A) = 0$ or $y(A) = 0$. Consider the following initial value problem

$$\begin{cases} \dot{x} & = & x(1-x) - \frac{\alpha x \tilde{y}}{x+\tilde{y}}, \\[2mm] \dot{\tilde{y}} & = & \tilde{y}\left(-\delta + \frac{\beta x(t-\tau)}{x(t-\tau)+\tilde{y}(t-\tau)} - \frac{h}{h+\tilde{y}-T}\right), \\[2mm] \tilde{y}(0) & = & y(0) > 0, \end{cases} \tag{2.12}$$

For any $t \in [-\tau, A)$, the following integral equation follows from (2.12)

$$\begin{cases} x(t) & = & x(0)\exp\int_0^t \left(1 - x(s) - \frac{\alpha\tilde{y}(s)}{x(s)+\tilde{y}(s)}\right)ds, \\[2mm] \tilde{y}(t) & = & y(0)\exp\int_0^t \left(-\delta + \frac{\beta x(s-\tau)}{x(s-\tau)+\tilde{y}(s-\tau)} - \frac{h}{h+\tilde{y}-T}\right)ds. \end{cases} \tag{2.13}$$

From the continuity of $x(t)$ and $\tilde{y}(t)$ on $[-\tau, A)$ one can find a positive number M, such that for all $t \in [\tau, A)$,

$$\begin{cases} x(t) & = & x(0)\exp\int_0^t \left(1 - x(s) - \frac{\alpha\tilde{y}(s)}{x(s)+\tilde{y}(s)}\right)ds \geq x(0)e^{-TM}, \\[2mm] \tilde{y}(t) & = & y(0)\exp\int_0^t \left(-\delta + \frac{\beta x(s-\tau)}{x(s-\tau)+\tilde{y}(s-\tau)} - \frac{h}{h+\tilde{y}-T}\right)ds \geq y(0)e^{-TM}. \end{cases} \tag{2.14}$$

By standard comparison principal, we have $y(t) \geq \tilde{y}(t)$ for all $t \in [0, +\infty)$. So for all $t \in [\tau, A)$, we have

$$\begin{cases} x(t) & = & x(0)\exp\int_0^t \left(1 - x(s) - \frac{\alpha y(s)}{x(s)+y(s)}\right)ds \geq x(0)e^{-TM}, \\[2mm] y(t) & \geq & y(0)\exp\int_0^t \left(-\delta + \frac{\beta x(s-\tau)}{x(s-\tau)+\tilde{y}(s-\tau)} - \frac{h}{h+\tilde{y}-T}\right)ds \geq y(0)e^{-TM}. \end{cases} \tag{2.15}$$

Taking $t \to A$, one get $x(A) > 0$ and $y(A) > 0$, a contradiction. Thus the first quadrant is invariant for the system (2.1). $\square$

Lemma 2.7. *Let $(x(t), y(t))$ be the solution of (2.1). If $\beta > \delta$, then*

$$\limsup_{t \to +\infty} x(t) \le 1,$$

$$\limsup_{t \to +\infty} y(t) \le (\frac{\beta - \delta}{\delta}) e^{\beta \tau}.$$

Proof. From the first equation of system (2.1), one get that for all $t \in [0, \infty)$

$$\dot{x}(t) \le x(t)(1 - x(t)).$$

Consider the following initial value problem

$$\begin{cases} \dot{\tilde{x}}(t) & = & \tilde{x}(t)(1 - \tilde{x}(t)), \\ \tilde{x}(0) & = & x(0) > 0. \end{cases} \tag{2.16}$$

By standard comparison principal, we have $x(t) \le \tilde{x}(t)$ for all $t \in [0, +\infty)$. Thus

$$\limsup_{t \to +\infty} x(t) \le \limsup_{t \to +\infty} \tilde{x}(t) = 1.$$

From the second equation, we have

$$\dot{y}(t) \le \beta y(t). \tag{2.17}$$

Thus

$$y(t) \le y(0) e^{\beta t}.$$

Thus for $t > \tau$, integrating (2.17) on $[t - \tau, t]$, we obtain

$$y(t - \tau) \ge y(t) e^{\beta \tau}.$$

Note that there exists $A > 0$ such that for all $t > A$, $x(t) < 1$. Hence for $t > A + \tau$,

$$\dot{y}(t) \le y(t) \left(\frac{\beta}{1 + y e^{-\beta \tau}} - \delta \right) - H(y) \le y(t) \left(\frac{\beta}{1 + y e^{-\beta \tau}} - \delta \right).$$

A standard comparison argument shows that

$$\limsup_{t \to +\infty} y(t) < \left(\frac{\beta - \delta}{\delta} \right) e^{\beta \tau}.$$

Hence if $\beta > \delta$, then the system is bounded. $\qquad \square$

Lemma 2.8. *If $\beta - \delta - \frac{hM}{h + M - T} > 0$ and $\alpha < 1$, then the system (2.1) has a positive equilibrium, where*

$$M = \max\{\limsup_{t \to +\infty} x(t), \limsup_{t \to +\infty} y(t)\}.$$

Proof. One can show easily that if $\alpha < 1$, then

$$\dot{x} > x(1 - x - \alpha),$$

which implies that

$$\liminf_{t \to +\infty} x(t) \ge (1 - \alpha).$$

Therefore for any $\nu > 1$, there exists a positive T_ν such that for $t > T_\nu$, $x(t) > \frac{1-\alpha}{\nu}$ and $y(t) < \nu M$. Thus for $t > T_\nu + \tau$, we have

$$\dot{y}(t) \geq y(t) \left(\frac{\beta \frac{1-\alpha}{\nu}}{\frac{1-\alpha}{\nu} + \nu M} - \delta - \frac{\nu h M}{h + \nu M - T} \right).$$

Hence

$$\dot{y}(t) > y(t) \left(-\delta - \frac{\nu h M}{h + \nu M - T} \right),$$

which implies that for $t > T_\nu + \tau$,

$$y(t - \tau) > y(t) e^{\left(\delta + \frac{\nu h M}{h + \nu M - T} \right)\tau}.$$

Thus for $t > T_\nu + \tau$

$$\dot{y}(t) \geq y(t) \left(\frac{\beta \frac{1-\alpha}{\nu}}{\frac{1-\alpha}{\nu} + y(t) e^{\left(\delta + \frac{\nu h M}{h + \nu M - T} \right)}} - \delta - \frac{\nu h M}{h + \nu M - T} \right).$$

which yields

$$\liminf_{t \to +\infty} y(t) \geq \left(\frac{\beta \frac{1-\alpha}{\nu}}{\delta + \frac{h M}{h + M - T}} - \frac{1-\alpha}{\nu} \right) e^{\left(\delta + \frac{\nu h M}{h + \nu M - T} \right)\tau}.$$

As $\nu \to 1$, we get

$$\liminf_{t \to +\infty} y(t) \geq \frac{\left(\beta - \delta - \frac{h M}{h + M - T} \right) (1 - \alpha)}{\delta + \frac{h M}{h + M - T}} e^{-\left(\delta + \frac{h M}{h + M - T} \right)\tau} > 0.$$

$\square$

Recall that system (2.1) is said to be not persistent, if

$$\min(\liminf_{t \to +\infty} x(t), \liminf_{t \to +\infty} y(t)) = 0,$$

for some of its positive solutions.

Lemma 2.9. *If $\alpha > 1 + \delta$, then the system (2.1) is not persistent.*

Proof. If $\alpha > 1 + \delta$, then there exists an $\epsilon > 0$ such that

$$\frac{\alpha}{1 + \epsilon} = 1 + \delta.$$

Let $\frac{x(0)}{y(0)} < \epsilon$, we claim that for all $t > 0$, $\frac{x(t)}{y(t)} < \epsilon$. Otherwise, there exists $t_0 > 0$ such that $\frac{x(t_0)}{y(t_0)} = \epsilon$ and for $t \in [0, t_0)$, $\frac{x(t)}{y(t)} < \epsilon$. Then for $t \in [0, t_0)$, we have

$$\dot{x}(t) \leq x(t) \left(1 - \frac{\alpha}{1 + \epsilon} \right),$$

from which we obtain

$$x(t) \leq x(0) e^{\left(1 - \frac{\alpha}{1 + \epsilon} \right)} = x(0) e^{-\delta t}.$$

Thus $x(t) \leq x(0) e^{-\delta t}$ for all $t \geq 0$. That is $\lim_{t \to +\infty} x(t) = 0$. Similarly,

$$\dot{y}(t) \geq -y(t) \left(\delta + \frac{h}{h + y - T} \right),$$

from which we obtain

$$y(t) \geq y(0)e^{-\delta t}.$$

Thus for $t \in [0, t_0]$,

$$\frac{x(t)}{y(t)} \leq \frac{x(0)e^{\left(1 - \frac{\alpha}{1+\epsilon}\right)}}{y(0)e^{-\delta t}} = \frac{x(0)}{y(0)} < \epsilon.$$

Hence the system is not persistent if $\alpha > 1 + \delta$. □

Theorem 2.10. *If $\alpha > 1 + \delta$ and $\beta < \frac{\alpha\delta}{\alpha - 1 - \delta}$, then there exists a positive solution $(x(t), y(t))$ such that $\lim_{t \to +\infty} (x(t), y(t)) = (0, 0)$.*

Proof. If $\alpha > 1 + \delta$, then $\lim_{t \to +\infty} x(t) = 0$ and for $t \geq 0$,

$$\frac{x(t)}{y(t)} \leq \frac{\alpha}{1 + \delta} - 1,$$

provided that

$$\frac{x(0)}{y(0)} \leq \frac{\alpha}{1 + \delta} - 1.$$

Hence for $t \geq \tau$,

$$\dot{y}(t) \leq y(t) \left(\frac{\beta}{1 + \frac{1+\delta}{\alpha - 1 - \delta}} - \delta \right),$$

which implies

$$\lim_{t \to +\infty} y(t) = 0,$$

if $\beta < \frac{\alpha\delta}{\alpha - 1 - \delta}$. □

One concludes that under the assumption $\alpha > 1 + \delta$, system (2.1) may have positive steady state. This shows that system (2.1) can have both positive steady state and positive solutions that tend to the origin.

3. Stability of the equilibria of the model without time delay

In this section, we study the local behavior of the model around its equilibria. The general Jacobian matrix of system (2.1) without delay around an arbitrary point (x, y) equals

$$J = \begin{pmatrix} 1 - 2x - \frac{\alpha y^2}{(x+y)^2} & -\frac{\alpha x^2}{(x+y)^2} \\ \frac{\beta y^2}{(x+y)^2} & -\delta + \frac{\beta x^2}{(x+y)^2} - \frac{dH(y)}{dy} \end{pmatrix}, \tag{3.1}$$

where

$$\frac{dH(y)}{dy} = \begin{cases} 0 & 0 < y \leq T, \\ \frac{h^2}{(h+y-T)^2} & y > T. \end{cases}$$

Here, we shall point out, although $(0, 0)$ is defined for system (2.1), it cannot be linearized at. So, local stability of $(0, 0)$ can not be studied. Indeed, this singularity at the origin, while causes much difficulty in our analysis of the system, contributes significantly to the richness of dynamics of the model. A complete parametric analysis

of stability properties and dynamic around the complicated equilibrium $(0,0)$ for unharvested model is done in [5]. Since $T > 0$, the results of [5] are valid for harvested model too.

From the expression (3.1) the following result can be proved immediately.

Theorem 3.1. *At the point $E = (1,0)$, the trace and determinant of (3.1) are $Tr(J)_{(1,0)} = -1 - \delta + \beta$ and $Det(J)_{(1,0)} = \delta - \beta$. Therefore*

(1) *if $\delta - \beta < 0$ then E is a saddle;*
(2) *if $\delta - \beta > 0$ then E is a stable node;*
(3) *if $\delta - \beta = 0$ then E remains a stable node.*

Now we study the linearized system at the interior equilibria $E^* = (x^*, y^*)$, $E^{**} = (x^{**}, y^{**})$.

Theorem 3.2. *Let*

$$M = \frac{(\beta - \delta)(-\alpha\delta\beta^2 + \alpha\delta^2\beta + \beta^2\delta)}{\beta^3},$$

$$N = \frac{-\beta^2 + \alpha(\beta^2 - \delta^2) - \beta\delta(\beta - \delta)}{\beta^2}.$$

If $y^* \leq T$, then we have

(1) *if $M < 0$, then E^* is a saddle;*
(2) *if $M > 0$ and $N < 0$, then E^* is a stable node or focus;*
(3) *if $M > 0$ and $N > 0$, then E^* is an unstable node or focus.*

Proof. The Jacobian matrix of system (2.1) at E^* is

$$J = \begin{pmatrix} \frac{-\beta^2 + \alpha(\beta^2 - \delta^2)}{\beta^2} & -\frac{\alpha\delta^2}{\beta^2} \\ \\ \frac{(\beta-\delta)^2}{\beta} & \frac{\delta(\delta-\beta)}{\beta} \end{pmatrix}.$$

The associated characteristic equation is

$$\lambda^2 - N\lambda + M = 0.$$

Thus the eigenvalues of the Jacobian matrix are

$$\lambda_{1,2} = \frac{N \pm \sqrt{N^2 - 4M}}{2},$$

and the result is obtained immediately. $\square$

Remark 3.3. By Eq. (2.6), if the interior equilibrium E^* exists, then M is positive. Thus the coexistence equilibrium E^* cannot be a saddle when it is biologically feasible.

Note that at the equilibrium (x^{**}, y^{**}) the trace and the determinant of the Jacobian matrix equals

$$Tr(J) = C - \frac{B^2}{\alpha} - \delta + \frac{\beta A^2}{\alpha^2} - \phi,$$

$$Det(J) = C(\frac{\beta}{\alpha^2}A^2 - \phi - \delta) + \frac{1}{\alpha}B^2(\delta + \phi),$$

where $\phi = \frac{h^2}{h - T - \frac{x^{**}B}{A}}$, $A = 1 - \alpha - x^{**}$, $B = x^{**} - 1$, $C = 1 - 2x^{**}$.

Theorem 3.4. *By the above mentioned notations we have*

(1) *if $C - \frac{1}{\alpha}B^2 > \frac{\beta C A^2}{\alpha^2(\phi+\delta)}$, then E^{**} is a saddle point;*

(2) *if $C - \frac{1}{\alpha}B^2 < \frac{\beta C A^2}{\alpha^2(\phi+\delta)}$ and $C - \frac{1}{\alpha}B^2 < \delta + \phi - \frac{\beta A^2}{\alpha^2}$, then E^{**} is a stable node or focus;*

(3) *if $\delta + \phi - \frac{\beta A^2}{\alpha^2} < C - \frac{1}{\alpha}B^2 < \frac{\beta C A^2}{\alpha^2(\phi+\delta)}$, then E^{**} is a unstable node or focus.*

4. Stabilization effect of the delay

In this section, we study the effect of time delay on the stability of the co-existence equilibrium of the system. The linearized system at the equilibrium (x_0, y_0) equals

$$\begin{cases} u'(t) & = & a_{11}u(t) + a_{12}v(t), \\ v'(t) & = & a_{21}u(t-\tau) + a_{22}v(t-\tau) - \left(\delta + \frac{dH(y)}{dy}\right)v(t), \end{cases} \tag{4.1}$$

where $u(t) = x(t) - x_0$, $v(t) = y(t) - y_0$,

$$a_{11} = 1 - 2x_0 - \frac{\alpha y_0^2}{(x_0 + y_0)^2}, \quad a_{12} = -\frac{\alpha x_0^2}{(x_0 + y_0)^2},$$

$$a_{21} = \frac{\beta y_0^2}{(x_0 + y_0)^2}, \quad a_{22} = \frac{\beta x_0^2}{(x_0 + y_0)^2}.$$

Denote system (4.1) by

$$\dot{X}(t) = A_0 X(t) + A_1 X(t - \tau), \tag{4.2}$$

where

$$A_0 = \begin{bmatrix} a_{11} & a_{12} \\ 0 & -\delta - \frac{dH(y)}{d(y)} \end{bmatrix},$$

and

$$A_1 = \begin{bmatrix} 0 & 0 \\ a_{21} & a_{22} \end{bmatrix},$$

and $X(t) = [x(t), y(t)]^t$. The characteristic equation of (4.1) is

$$\lambda^2 + A\lambda + B + (C\lambda + D)e^{-\lambda\tau} = 0, \tag{4.3}$$

where

$$A = -a_{11} + \left(\delta + \frac{dH(y)}{dy}\right), \quad B = -a_{11}\left(\delta + \frac{dH(y)}{dy}\right),$$

$$C = -a_{22}, \quad D = a_{11}a_{22} - a_{12}a_{21}.$$

When $\tau = 0$, Eq. (4.3) becomes

$$\lambda^2 + (A + C)\lambda + (B + D). \tag{4.4}$$

All roots of Eq. (4.4) have negative real parts if and only if

    (C1): $A + C > 0$,

(C2): $B + D > 0$.

Recall that the equilibrium (x_0, y_0) is called absolutely stable if it is asymptotically stable for all delays $\tau \geq 0$, and is called conditionally stable if it is asymptotically stable for τ in some intervals, but not necessarily for all delays $\tau \geq 0$.

Lemma 4.1. [7] *System (4.2) is absolutely stable if and only if*
(1) $Re\lambda(A_0 + A_1) < 0$;
(2) $det[i\omega I - A_0 - A_1 e^{-i\omega\tau}] \neq 0$ *for all* $\omega > 0$.

By the first assumption of above lemma, system (4.2) with $\tau = 0$ is asymptotically stable, while assumption (2) ensures that $i\omega$ is not a root of Eq. (4.3). Thus, roughly speaking, Lemma 4.1 says that the delayed system (4.2) is absolutely stable if and only if the corresponding ODE system is asymptotically stable and the characteristic Eq. (4.3) has no purely imaginary roots. Lemma 4.1 will be used to study stability and bifurcation in various delayed systems. The main idea is as follows. If assumption (2) does not hold, that is, if the characteristic Eq. (4.3) has a pair of purely imaginary roots, say $\pm i\omega_0$ then system (4.2) is not absolutely stable but can be conditionally stable. Suppose ω_0 is achieved when τ reaches a value τ^0. When $\tau < \tau^0$ the real parts of all roots of the characteristic Eq. (4.3) still remain negative and system (4.2) is conditionally stable. When $\tau = \tau^0$, the characteristic Eq. (4.3) has a pair of purely imaginary roots $\pm i\omega_0$ and system (4.2) loses its stability. By Rouchés theorem [8] and continuity, if the transversality condition holds at $\tau = \tau^0$ then when $\tau > \tau^0$ the characteristic Eq. (4.3) will have at least one root with positive real part and system (4.2) becomes unstable. Moreover, Hopf bifurcation occurs, that is, a family of periodic solutions bifurcates from the steady state as τ passes through the critical value τ^0.

Let $\lambda = i\omega, \omega > 0$ be the root of the characteristic equation. Then we have

$$-C\omega \sin \tau\omega - D \cos \tau\omega = -\omega^2 + B, \tag{4.5}$$

$$-C\omega \cos \tau\omega + D \sin \tau\omega = A\omega. \tag{4.6}$$

From this it follows that

$$\omega^4 - (C^2 - A^2 + 2B)\omega^2 + B^2 - D^2 = 0. \tag{4.7}$$

Eq. (4.7) has two roots

$$\omega_\pm = \frac{(C^2 - A^2 + 2B) \pm \sqrt{(C^2 - A^2 + 2B)^2 - 4(B^2 - D^2)}}{2}.$$

If

(C3): $C^2 - A^2 + 2B < 0$, $B^2 - D^2 > 0$ or $C^2 - A^2 + 2B < 4(B^2 - D^2)$,

then none of ω_+ and ω_- is positive. That is Eq. (4.7) does not have positive roots. Therefore characteristic Eq. (4.3) does not have purely imaginary roots. Since (C1) and (C2) ensure that all roots of Eq. (4.4) have negative real parts, by Rouché's theorem it follows that the roots of Eq. (4.3) have negative real roots for all $\tau \geq 0$.

On the other hand if

(C4): $B^2 - D^2 < 0$ or $C^2 - A^2 + 2B > 0$ and $(C^2 - A^2 + 2B)^2 = 4(B^2 - D^2)$, then Eq. (4.7) has a positive root ω_+. If

(C5): $B^2 - D^2 > 0$, $C^2 - A^2 + 2B > 0$ and $(C^2 - A^2 + 2B)^2 > 4(B^2 - D^2)$, then Eq. (4.7) has two positive root $\omega_\pm$. In both cases the characteristic Eq. (4.3) has purely imaginary roots when τ takes certain values.

From Eq. (4.5) and (4.6) the corresponding critical time delay is given by

$$\tau^\pm = \frac{1}{\omega_\pm} \arccos\left\{ \frac{D(\omega_\pm^2 - B) - AC\omega_\pm^2}{C^2\omega_\pm^2 + D^2} \right\} + \frac{2j\pi}{\omega_\pm}, \ j = 0, 1, 2, \ldots. \tag{4.8}$$

The above analysis can be summarized into following lemma.

Lemma 4.2.
(1) *If (C1), (C2) and (C4) hold on and $\tau = \tau^+$, then Eq. (4.3) has a pair of purely imaginary roots $\pm i\omega_+$.*
(2) *If (C1), (C2) and (C5) hold on and $\tau = \tau^+$ ($\tau = \tau^-$ resp.), then Eq. (4.3) has a pair of conjugate imaginary roots $\pm i\omega_+$ ($\pm i\omega_-$ resp.).*

Let $\lambda^\pm = \alpha^\pm + i\omega^\pm$ be the roots of Eq. (4.3) satisfying

$$\alpha^\pm(\tau^\pm) = 0, \omega^\pm(\tau^\pm) = \omega_\pm.$$

Differentiating (4.3) with respect to τ and substituting $\tau = \tau^\pm$, one can verify that the following transversality conditions hold

$$\frac{d}{d\tau} \operatorname{Re} \lambda^+(\tau^+) > 0, \frac{d}{d\tau} \operatorname{Re} \lambda^-(\tau^-) < 0.$$

It follows that $\tau^\pm$ are bifurcation values. Thus we have the following theorem.

Theorem 4.3. *Let $\tau^\pm$ be defined by (4.8), then we have*
(1) *If (C1)-(C3) hold, then all roots of Eq. (4.3) have negative real parts for all $\tau \geq 0$.*
(2) *If (C1), (C2) and (C4) hold, then for $\tau \in [0, \tau^+)$ all roots of Eq. (4.3) have negative real parts. When $\tau = \tau^+$, Eq. (4.3) has a pair of purely imaginary roots $\pm i\omega_+$ and when $\tau > \tau^+$, then Eq. (4.3) has at least one root with positive real part.*
(3) *If (C1), (C2) and (C5) hold, then for $\tau \in [0, \tau^+) \cup (\tau^-, +\infty)$ all roots of Eq. (4.3) have negative real parts and when $\tau \in [\tau^+, \tau^-]$, Eq. (4.3) has at least one root with positive real part.*

5. Numerical simulations

In this section, we present some numerical simulations to illustrate our theoretical analysis. In the first example we show that when the time delay passes the critical time, the system experiences the periodic behavior.

Example 5.1. *Consider the unharvested model (2.1), with the parameter values $\alpha = 1.3, \beta = 0.8, \delta = 0.4$. As shown in the FIGURE 2, by (2.5), the system has an interior equilibrium at $(0.35, 0.35)$. By direct calculations, from the relations in Section 4, one get $a_{11} = -0.025, a_{12} = -0.325, a_{21} = 0.26, a_{22} = 0.26$. Thus $A = 0.425, B =$*

FIGURE 2. $\alpha = 1.3, \beta = 0.8, \delta = 0.4$, *without harvesting, without delay.*
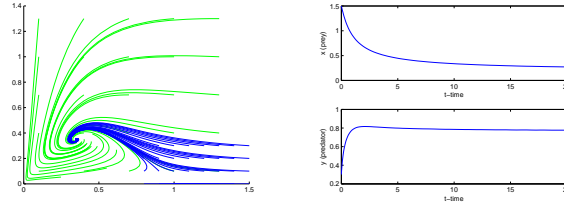

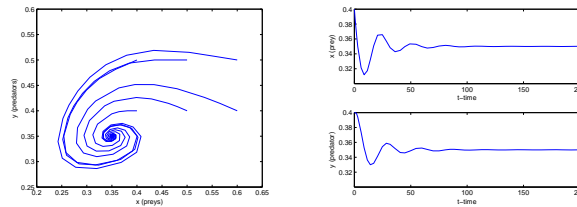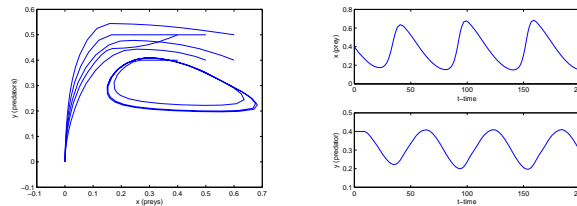FIGURE 3. $\alpha = 1.3, \beta = 0.8, \delta = 0.4$, *without harvesting,* $\tau = 1$.
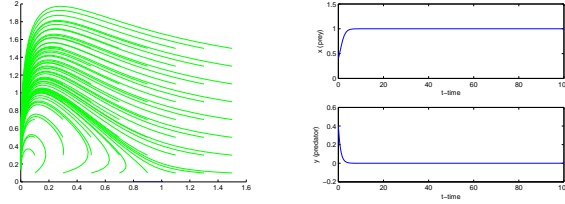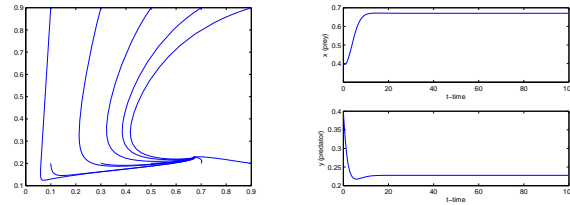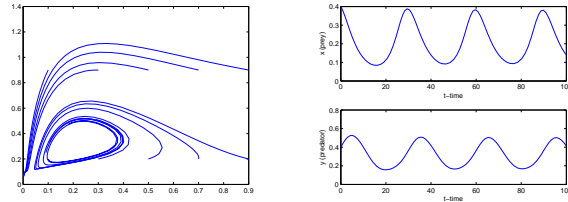

FIGURE 4. $\alpha = 1.3, \beta = 0.8, \delta = 0.4$, *without harvesting,* $\tau = 9$.

$0.1, C = -0.26, D = 0.078$. *We have* $A + C > 0$ *and* $B + D > 0$ *and* $B^2 - D^2 < 0$, *so the conditions (C1), (C2) and (C4) hold on. Hence Eq. (4.7) has a positive root* $\omega^+ = 0.202492344$ *and the critical time delay is* $\tau^+ = \frac{1}{\omega^+} * arccos(0.75668594) \simeq 3.5$. *In FIGURE 3, the phase portrait of the unharvested system with time delay* $\tau = 1$ *is shown. In FIGURE 4, the phase portrait of the unharvested system with time delay* $\tau = 9$ *is shown. Since* $\tau^+ < 9$ *the system undergoes the oscillatory behavior.*

In the second example the harvesting creates a co-existence equilibrium when it does not exist in unharvested model. By Theorem 2.3 we know that if $\beta > \delta$, $\beta - \alpha\beta + \alpha\delta \leq 0$, then the system (2.1) has no interior equilibrium and the extinction of the species is inevitable.

FIGURE 5. $\alpha = 1.3, \beta = 0.8, \delta = 0.1$, without harvesting.

FIGURE 6. $\alpha = 1.3, \beta = 0.8, \delta = 0.1, T = 0.1, h = 1$.

FIGURE 7. $\alpha = 1.3, \beta = 0.8, \delta = 0.1, T = 0.1, h = 0.1$.

Example 5.2. *In FIGURE 5, the phase portrait of the system with the parameter values $\alpha = 1.3, \beta = 0.8, \delta = 0.1$ without harvesting has been shown. The system has no co-existence equilibria since $\beta - \alpha\beta + \alpha\delta < 0$. Then in FIGURE 6, the threshold harvesting function with the parameter values $h = 1, T = 0.1$ is added to the system. In this case, the system has a stable interior equilibrium. Finally in FIGURE 7, the system experience periodic behavior with the harvesting parameters $h = 0.1, T = 0.1$.*

6. Conclusions

We study a ratio-dependent functional response predator-prey model with a threshold harvesting and with a time delay in the predator equation. We study the equilibria of the system before and after the threshold. Furthermore we show that the threshold harvesting can improve the undesirable behavior such as nonexistence of interior

equilibria. The system undergoes a Hopf bifurcation when it passes a critical time delay.

7. Acknowledgment

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