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Numerical solution of Troesch's problem using Christov rational functions

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Abstract We present a collocation method to obtain the approximate solution of Troesch's problem which arises in the confinement of a plasma column by radiation pressure and applied physics. By using the Christov rational functions and collocation points, this method transforms Troesch's problem into a system of nonlinear algebraic equations. The rate of convergence is shown to be exponential. The numerical results obtained by the present method compares favorably with those obtained by various methods earlier in the literature.

Keywords. Troesch's problem, Christov functions, Collocation, Wiener functions.

2010 Mathematics Subject Classification. 65L60, 41A20.

1. Introduction

The aim of this paper is to introduce a new approach for the numerical solution of the Troesch's problem. This problem, which is a nonlinear two point boundary-value problem, is given by

$$y'' = \lambda \sinh(\lambda y), \qquad 0 \le x \le 1, \tag{1.1}$$

subject to the boundary conditions

$$y(0) = 0, \quad y(1) = 1, \tag{1.2}$$

where λ is a positive constant. Troesch's problem arises from a system of nonlinear ordinary differential equations which occur in an investigation of the confinement of a plasma column by radiation pressure [28]. Also, Troesch's problem occurs in the

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theory of gas porous electrodes [15]. Roberts and Shipman [23] give the closed form solution to this problem in terms of the Jacobian elliptic function:

$$y(x) = \frac{2}{\lambda} \sinh^{-1} \left\{ \frac{y'(0)}{2} \operatorname{sc} \left(\lambda x | 1 - \frac{1}{4} y'(0)^2 \right) \right\}, \tag{1.3}$$

where $y'(0) = 2\sqrt{1-m}$, and the constant m satisfies the solution of the transcendental equation

$$\frac{\sinh\left(\frac{\lambda}{2}\right)}{\sqrt{1-m}} = \operatorname{sc}(\lambda|m). \tag{1.4}$$

Here, the Jacobian elliptic function $\operatorname{sc}(\lambda|m)$ is defined by $\operatorname{sc}(\lambda|m) = \tan\phi$, where ϕ, λ and m are related by the integral

$$\lambda = \int_0^{\phi} \frac{1}{\sqrt{1 - m\sin^2\theta}} d\theta.$$

It has been shown that $y(x)$ has a singularity located at a pole of $\operatorname{sc}(\lambda|m)$ or approximately at [23, 27]

$$x_s = \frac{1}{\lambda} \ln\left(\frac{8}{y'(0)} \right). \tag{1.5}$$

As pointed by [23], in addition to its intrinsic interest, Troesch's problem has become something of a test case for methods of solving unstable two-point boundary value problems because of its difficulties.

There are different techniques for solving Troesch's problem. Temimi [25] proposed a new discontinuous Galerkin finite element method to solve this problem. Chang in [6] used the simple shooting method and in [5] the author applied the variational iteration method for solving Troesch's problem. Also, a numerical algorithm based on the decomposition method is presented by Deeba et al. [12] for this problem. In [32] the sinc-Galerkin method is used to solve the nonlinear two point boundary value problem with application to Troesch's equation. Khuri and Sayfy [20] used a finite element approach based on the cubic B-spline collocation method on both a uniform mesh and a piecewise-uniform Shishkin mesh to solve this problem. Moreover, the modified homotopy perturbation technique [14], differential transform method [7], reproducing kernel Hilbert space method [17] and the Jacobi collocation method [13] are employed to solve Troesch's problem.

In the current investigation, we construct the solution of Troesch's problem using a different approach. This approach is based on the collocation technique. Our method consists of reducing the problem to the solution of algebraic equations by expanding the required approximate solution as the elements of the Christov rational functions (CRFs) with unknown coefficients. The properties of CRFs are then utilized to evaluate the unknown coefficients.

The current paper is organized as follows: Section 2 is devoted to the basic formulation of CRFs required for our subsequent development. Section 3 applies the collocation method together with CRFs to the Troesch's problem. In Section 4, we present some numerical examples to show the efficiently and applicability of this method. An application of the model is described in Section 5. Also a conclusion is given in Section 6.

2. Properties of the CRFs

Norbert Wiener introduces the complex-valued rational functions [31, page 35]

$$\rho_n(x) = \frac{1}{\sqrt{\pi}} \frac{(ix+1)^n}{(ix-1)^{(n+1)}}, \quad n = 0, 1, 2, ..., \; i = \sqrt{-1}, \tag{2.1}$$

as Fourier transforms of the Laguerre functions. Higgins [16] defined it also for negative indices n and proved its completeness and orthogonality. One way to see this is to make the change of variable [30]

$$e^{i\theta} = \frac{1+ix}{1-ix}, \quad \text{i.e.} \quad x = \tan\frac{\theta}{2},$$

which maps the entire real line $x \in [-\infty, \infty]$ to $\theta \in [-\pi, \pi]$. The first work which exclusively deals with the basis functions (2.1) appears to come from [10] in 1982. Christov [10] show that this system consists of two real subsequences of odd functions S_n and even functions C_n, namely

$$S_n(x) = \frac{\rho_n(x) + \rho_{-n-1}(x)}{i\sqrt{2}}, \quad n = 0, 1, 2, ..., \tag{2.2}$$

$$C_n(x) = \frac{\rho_n(x) - \rho_{-n-1}(x)}{\sqrt{2}}, \quad n = 0, 1, 2, \dots . \tag{2.3}$$

As pointed by [10], both sequences are orthonormal and each member of (2.2) is orthogonal to all members of (2.3); each member of (2.3) is also orthogonal to all members of (2.2). It is worth indicating that, S_n and C_n can be defined for negative n through the relations

$$S_{-n} = S_n \quad \text{and} \quad C_{-n} = C_n.$$

Unlike Hermite and Laguerre sets of functions which behave exponentially at infinity, this system exhibits asymptotic behavior x^{-1} for the odd sequence and x^{-2} for the even one [10]. The Weiner functions applied successfully to time integration of the Benjamin-Ono equation by James and Weideman [18]. The functions S_n and C_n have been employed as forward solvers in the solution of a differential equation on an infinite interval [10], KdV and Kuramoto-Sivashinsky equation [11], fifth order KdV [1], the time dependent problem of interacting 1D solitons [8] and for computing the stationary solutions of Boussinesq equation [9]. We also refer the interested reader to [2–4]. Very recently, Narayan and Hesthaven [21, 22] introduced the generalized Wiener rational basis functions and generalized the algebraically-decaying functions of Wiener. Most of the practically important formulae for the functions S_n and C_n are discussed thoroughly in [10] and here an overview of the basic formulation of these functions required for our subsequent development is presented. The odd functions S_n and even functions C_n can be expressed in an explicit way:

$$S_n(x) = \sqrt{\frac{2}{\pi}} \frac{\sum_{k=1}^{n+1} (-1)^{n+k} \binom{2n+1}{2k-1} x^{2k-1}}{(x^2+1)^{n+1}}, \quad n = 0, 1, 2, ..., \tag{2.4}$$

$$C_n(x) = \sqrt{\frac{2}{\pi}} \frac{\sum_{k=1}^{n+1}(-1)^{n+k+1}\binom{2n+1}{2k-2}x^{2k-2}}{(x^2+1)^{n+1}}, \quad n = 0, 1, 2, \dots. \tag{2.5}$$

Theorem 2.1. *For the first and second derivative of the basis functions, the following recurrence relation hold*

$$\frac{dS_n(x)}{dx} = \frac{1}{2}\Big\{nC_{n-1}(x) - (2n+1)C_n(x) + (n+1)C_{n+1}(x)\Big\}, \tag{2.6}$$

$$\frac{dC_n(x)}{dx} = \frac{-1}{2}\Big\{nS_{n-1}(x) - (2n+1)S_n(x) + (n+1)S_{n+1}(x)\Big\}, \tag{2.7}$$

$$\begin{aligned}\frac{d^2S_n(x)}{dx^2} = \frac{-1}{4}\Big\{&(n^2-n)S_{n-2}(x) - 4n^2S_{n-1}(x) + (6n^2+6n+2)S_n(x)\\ &- 4(n+1)^2S_{n+1}(x) + (n^2+3n+2)S_{n+2}(x)\Big\},\end{aligned} \tag{2.8}$$

$$\begin{aligned}\frac{d^2C_n(x)}{dx^2} = \frac{-1}{4}\Big\{&(n^2-n)C_{n-2}(x) - 4n^2C_{n-1}(x) + (6n^2+6n+2)C_n(x)\\ &- 4(n+1)^2C_{n+1}(x) + (n^2+3n+2)C_{n+2}(x)\Big\}.\end{aligned} \tag{2.9}$$

Proof. The proof is done by direct verification and the detailed proof can be found in [10]. □

Theorem 2.2. *The functions S_n and C_n can be expressed, by using trigonometric functions, in an explicit way:*

$$S_n(x) = (-1)^{n+1}\frac{\sin(n+1)\theta + \sin(n\theta)}{\sqrt{2}}, \tag{2.10}$$

$$C_n(x) = (-1)^{n}\frac{\cos(n+1)\theta + \cos(n\theta)}{\sqrt{2}}, \tag{2.11}$$

where $x = \tan\frac{\theta}{2}$ or $\theta = 2\arctan(x)$ is a transformation of the independent variable.

Proof. See Ref. [8]. □

As said in [8,9] any function $f(x)$ is a periodic function of θ with period 2π. Now, any localized function of $f(x)$ may be expanded as

$$f(x) = \sum_{n=0}^{\infty}(a_nC_n(x) + b_nS_n(x)). \tag{2.12}$$

which in its turn can be rewritten as a Fourier series for the periodic function

$$\begin{aligned}f\left(\tan\frac{\theta}{2}\right) = \sum_{n=0}^{\infty}\Bigg(&a_n(-1)^n\frac{\cos(n+1)\theta + \cos(n\theta)}{\sqrt{2}}\\ &+b_n(-1)^{n+1}\frac{\sin(n+1)\theta + \sin(n\theta)}{\sqrt{2}}\Bigg),\end{aligned} \tag{2.13}$$

which are known to have exponential convergence. Since, the Fourier series have exponential convergence for periodic functions, then the exponential convergence of C_n, S_n series follows [9].

3. Solution with CRFs

The basic idea of our method for solving Troesch's problem on the interval $[0,1]$ is to transform the problem, with a properly selected variable transformation, to the interval $(-\infty,+\infty)$ and then to solve the transformed problem by the collocation method. Let $y(x)$ be the solution of boundary value problem (1.1)-(1.2). First of all, we reformulate the problem by applying the transformation $\tilde{y}(x)=y(x)-x$ that makes the boundary conditions become homogeneous. Therefore problem (1.1)-(1.2) reduces to the following boundary value problem

$$\tilde{y}''(x)=\lambda\sinh(\lambda(\tilde{y}(x)+x)), \qquad 0\le x\le 1, \tag{3.1}$$

$$\tilde{y}(0)=0, \quad \tilde{y}(1)=0. \tag{3.2}$$

Now, application of the variable transformation [24]

$$x=\psi(t)=\frac{1}{2}\tanh\left(\frac{t}{2}\right)+\frac{1}{2}, \tag{3.3}$$

together with the change of notation

$$u(t)=\tilde{y}(\psi(t)), \tag{3.4}$$

transforms the problem to

$$u''(t)-\frac{\psi''(t)}{\psi'(t)}u'(t)=(\psi'(t))^2\lambda\sinh\Big(\lambda(u(t)+\psi(t))\Big), \qquad -\infty<t<+\infty, \tag{3.5}$$

subject to the boundary conditions

$$\lim_{t\to\pm\infty}u(t)=0. \tag{3.6}$$

Using (3.3) we get

$$\psi'(t)=\psi(t)(1-\psi(t)), \quad -\frac{\psi''(t)}{\psi'(t)}=2\psi(t)-1. \tag{3.7}$$

Substituting (3.7) in (3.5) we obtain

$$u''(t)+(2\psi(t)-1)u'(t)-(\psi(t))^2(1-\psi(t))^2\lambda\sinh\Big(\lambda(u(t)+\psi(t))\Big)=0, \tag{3.8}$$

We are now ready to solve (3.8) by using CRFs. The approximate solution for $u(t)$ in (3.8) is represented by

$$u(t)\approx u_N(t)=\sum_{i=0}^{N}(a_iC_i(t)+b_iS_i(t)). \tag{3.9}$$

It is noted that, the approximate solution in (3.9) satisfies the boundary conditions (3.6) since $S_n(t)$ and $C_n(t)$ are zero when $t \to \pm\infty$. A collocation scheme is defined by substituting $u_N(t)$ into the (3.8) and evaluating the result at

$$t_i = \ln\Big(\frac{x_i}{1-x_i}\Big), \qquad i = 1, 2, \cdots, 2N+2,$$

where

$$x_i = ih, \qquad i = 1, 2, \cdots, 2N+2, \qquad h = \frac{1}{2N+3}.$$

Employing Theorem 2.1, for $j = 1, 2, \cdots 2N+2$, we get

$$\Big(\sum_{i=0}^{N} a_i C_i''(t_j) + b_i S_i''(t_j)\Big) + (2\psi(t_j) - 1)\Big(\sum_{i=0}^{N} a_i C_i'(t_j) + b_i S_i'(t_j)\Big) \\ - (\psi(t_j))^2(1-\psi(t_j))^2 \lambda \sinh\Big(\lambda\Big(\Big(\sum_{i=0}^{N} a_i C_i(t_j) + b_i S_i(t_j)\Big) + \psi(t_j)\Big)\Big) = 0. \tag{3.10}$$

Equation (3.10) generates a set of $2N+2$ nonlinear algebraic equations, which can be solved for the unknown coefficients $\{a_i\}_{i=0}^N$ and $\{b_i\}_{i=0}^N$. Thus, $u_N(t)$ given in (3.9) can be calculated. Consequently the unknown function $y(x)$ may be approximated by

$$y_N(x) = u_N\left(2\tanh^{-1}(2x-1)\right) + x.$$

Remark 3.1. It is worth to mention here that, the system (3.10) can be solved by applying an iterative method, like the Newton's method. Having solved the nonlinear system (3.10) analytically for very small N, say $N = 1$, we see that some components of solution of this system are close to zero. Therefore, for larger values of N the initial guesses for a_i's and b_i's of the Newton's method can be estimated by values close to zero. Throughout this paper, we use the Maple's *fsolve* command with the initial approximation $a_i = b_i = 0.01$. Also, we note that the Newton's method has a convergence rate of quadratic order, which directly depends on the equations of (3.10) and initial guesses for a_i's and b_i's.

4. Results and discussion

To illustrate the effectiveness of our method we shall consider Troesch's problem for different values of the parameter λ. We calculated the approximate solution at the uniform grid points $x_i = ih, \quad i = 1, 2, \cdots 9,\ h = 0.1$. Taking $\lambda = 0.5, 1$, in Tables 1,2 and Figures 1 and 2 we compare absolute errors of the new method together with the results obtained by using the second-order modified homotopy perturbation method (MHPM) given in [14], cubic B-spline collocation method given in [20], sinc-Galerkin method (SGM) given in [32] and the one given in [19], which is based on Laplace transformation and a modified decomposition. From these tables and figures, we can see that CRFs solutions are either equally good or are better than those obtained by other methods. Also, Figures 3 and 4 displays solutions of the collocation method by using CRFs for different values of λ. It is clear from Figures 3 and 4 that solutions exhibit a boundary layer near $x = 1$ and gets narrower with increasing λ.

Table 1. Comparison of absolute error for $\lambda = 0.5$.

x	Exact	Laplace [19]	MHPM [14]	Spline [20]	SGM [32]	CRFs (N=9)
0.1	0.095176	7.7×10^{-4}	8.2×10^{-4}	7.7×10^{-4}	7.7×10^{-4}	2.1×10^{-4}
0.2	0.190633	1.5×10^{-3}	1.6×10^{-3}	1.5×10^{-3}	1.5×10^{-3}	5.8×10^{-4}
0.3	0.286653	2.1×10^{-3}	2.3×10^{-3}	2.1×10^{-3}	2.1×10^{-3}	1.2×10^{-3}
0.4	0.383522	2.7×10^{-3}	2.9×10^{-3}	2.7×10^{-3}	2.7×10^{-3}	1.9×10^{-3}
0.5	0.481537	3.0×10^{-3}	3.2×10^{-3}	3.0×10^{-3}	3.0×10^{-3}	2.4×10^{-3}
0.6	0.581002	3.1×10^{-3}	3.4×10^{-3}	3.1×10^{-3}	3.1×10^{-3}	2.5×10^{-3}
0.7	0.682235	3.0×10^{-3}	3.2×10^{-3}	3.0×10^{-3}	3.0×10^{-3}	2.3×10^{-3}
0.8	0.785571	2.4×10^{-3}	2.7×10^{-3}	2.4×10^{-3}	2.4×10^{-3}	1.8×10^{-3}
0.9	0.891367	1.5×10^{-3}	1.6×10^{-3}	1.5×10^{-3}	1.5×10^{-3}	9.3×10^{-4}

Table 2. Comparison of absolute error for $\lambda = 1$.

x	Exact	Laplace [19]	MHPM [14]	Spline [20]	SGM [32]	CRFs (N=6)
0.1	0.081797	2.9×10^{-3}	3.6×10^{-3}	2.8×10^{-3}	2.9×10^{-3}	4.8×10^{-5}
0.2	0.164530	5.9×10^{-3}	7.1×10^{-2}	5.6×10^{-3}	5.6×10^{-3}	2.3×10^{-3}
0.3	0.249167	8.2×10^{-3}	1.0×10^{-2}	8.2×10^{-3}	8.2×10^{-3}	4.2×10^{-3}
0.4	0.336732	1.0×10^{-2}	1.3×10^{-2}	1.0×10^{-2}	1.0×10^{-2}	5.7×10^{-3}
0.5	0.428347	1.2×10^{-2}	1.6×10^{-2}	1.2×10^{-2}	1.2×10^{-2}	7.7×10^{-3}
0.6	0.525274	1.3×10^{-2}	1.7×10^{-2}	1.3×10^{-2}	1.3×10^{-2}	6.9×10^{-3}
0.7	0.628971	1.3×10^{-2}	1.7×10^{-2}	1.3×10^{-2}	1.3×10^{-2}	6.0×10^{-3}
0.8	0.741168	1.1×10^{-2}	1.5×10^{-2}	1.1×10^{-2}	1.1×10^{-2}	7.7×10^{-3}
0.9	0.863970	7.4×10^{-3}	9.7×10^{-3}	7.4×10^{-3}	7.4×10^{-3}	1.8×10^{-3}

Figure 1. Comparison of absolute error for $\lambda = 0.5$

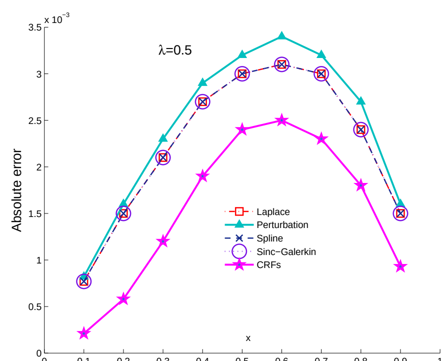


FIGURE 2. Comparison of absolute error for $\lambda = 1$

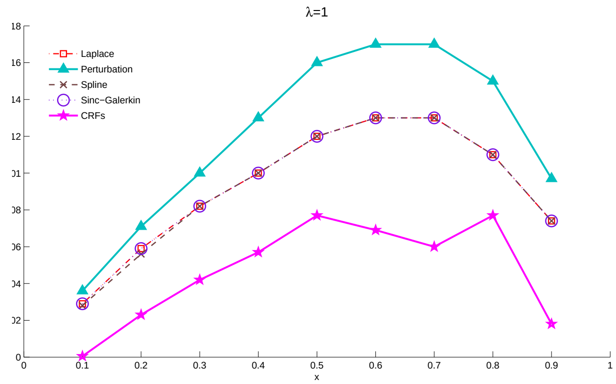
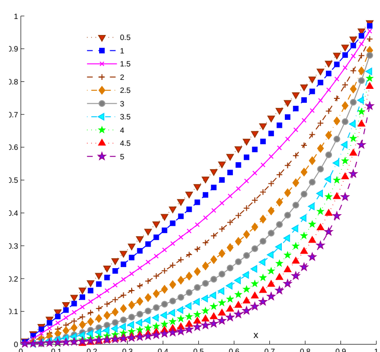


FIGURE 3. Aproximated solutions of Troesch's problem by using CRFs for $\lambda \leq 5$



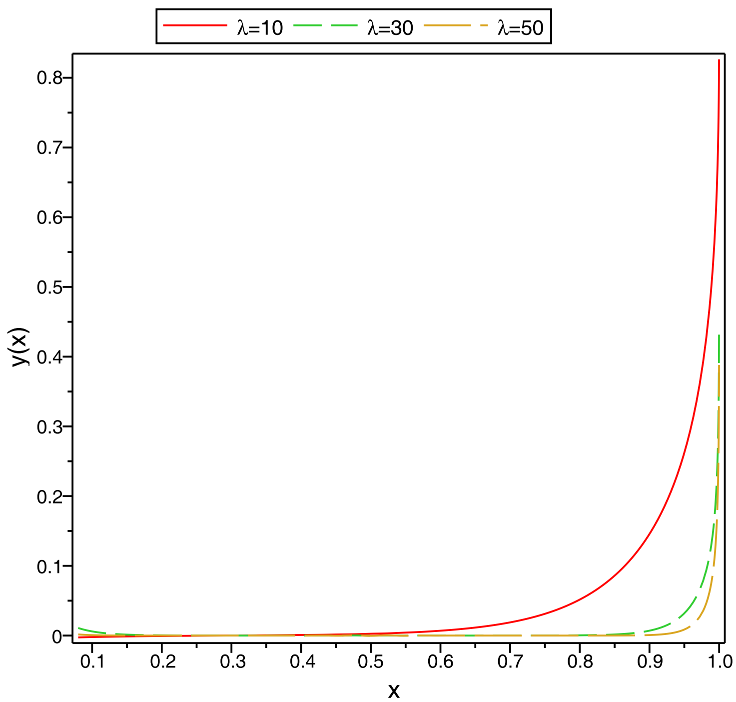
5. An application

The plasma will be regarded as consisting of two oppositely charged ideal gases which penetrate each other without friction. The Troesch's problem which arises in the investigation of the confinement of a plasma column by radiation pressure, was initially introduced and formulated by Weibel and Landshoff [29] and Troesch [26]. In this section we will trace its origin to a system of ordinary differential equations derived and solved in natural units [26, 29].

$$\frac{1}{r}\frac{d}{dr}(r\frac{dE_0}{dr}) + \left(\omega^2 - \frac{e^2 N}{M} - \frac{e^2 n}{m}\right) E_0 = 0, \tag{5.1}$$

$$\frac{1}{r}\frac{d}{dr}(rE_r) = e(N - n), \tag{5.2}$$

FIGURE 4. Aproximated solutions of Troesch's problem by using CRFs for $\lambda > 5$



$$E_r = -\frac{dU}{dr}, \tag{5.3}$$

$$n(r) = n_0 \exp\left(\frac{eU}{kT} - \frac{e^2 E_0^2}{4m\omega^2 kT}\right), \tag{5.4}$$

$$N(r) = N_0 \exp\left(-\frac{eU}{kT} - \frac{e^2 E_0^2}{4M\omega^2 kT}\right), \tag{5.5}$$

where n and N are variable ion and electron densities, $E = (E_r(r), 0, E_0(r)\cos\omega t)$ is the electric field and $E_0(r)\cos\omega t$ represents the applied field plus the field due to the plasma current. Also, T is temperature and equation (5.3) represents the radial electrostatic field due to charge separation. Moreover, Ion and electron temperatures are assumed equal and constant.

As described in [26], if these equations are considered in Cartesian, rather than polar coordinates, and if, E_0 is assumed to be negligibly small, then the system reduces to

$$\frac{dE}{dx} = N(x) - n(x), \tag{5.6}$$

$$E(x) = -\frac{dU}{dx}, \tag{5.7}$$

$$n(x) = n_0 \exp(\lambda U), \tag{5.8}$$

$$N(x) = N_0 \exp(-\lambda U), \tag{5.9}$$

up to constant factors. Substituting (5.7),(5.8) and (5.9) into (5.6), a second-order nonlinear ordinary differential equation is obtained as

$$\frac{d^2U}{dx^2} = N_0 \exp(-\lambda U) - n_0 \exp(\lambda U). \tag{5.10}$$

Also, applying the simplifying assumption $N_0 = n_0 = N^*$ and setting $U = -y$, we can write

$$y'' = 2N^* \sinh(\lambda y). \tag{5.11}$$

6. Conclusion

In this study, a new method using the Christov rational functions, to numerically solve the Troesch's problem is presented. The comparison of the results obtained by the present method, the exact solution and the other methods reveals that the method is very effective and convenient. The work emphasized our belief that the method is a reliable technique to handle these types of problems.

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