



## Iterative scheme to a coupled system of highly nonlinear fractional order differential equations

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**Abstract** In this article, we investigate sufficient conditions for existence of maximal and minimal solutions to a coupled system of highly nonlinear differential equations of fractional order with mixed type boundary conditions. To achieve this goal, we apply monotone iterative technique together with the method of upper and lower solutions. Also an error estimation is given to check the accuracy of the method. We provide an example to illustrate our main results.

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**Keywords.** Coupled system, Mixed type boundary conditions, Upper and lower solutions, Monotone iterative technique, Existence and uniqueness results.

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### 1. INTRODUCTION

In the last few decades, fractional calculus has attracted the attention of many researchers towards itself due to its many applications in various field of science and technology. It is mainly found that the tools of fractional calculus are more strong and more practical as compare to classical calculus in the mathematical modeling of various phenomena in the applied nature. These applications are found in various disciplines of science and engineering such as dynamics, aerodynamics, electrostatics, chemistry, biology, physics and biophysics, economics, control theory, signal and image processing, polymers rheology, thermodynamics and biomedical science, for more detail the reader should see the books [6, 9, 11, 12, 16, 18], and the references therein. Recently the area of fractional calculus involve differential equations of arbitrary order has gained considerable attention, because it has a lot of applications in almost every

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field of applied sciences, we refer to [3, 4, 23, 24, 27] etc, for some of its applications. The work related to the existence and uniqueness of positive solutions to differential equations has been studied by many researchers by means of some classical fixed point theorems, see [4, 7, 27, 28]. The researchers are taking interest in the study of coupled systems of boundary value problems for non linear fractional order differential equations and large number of research articles can be found in the literature dealing with existence and uniqueness of solutions, [1, 2, 8, 22] etc. For multiplicity of positive solutions, many techniques are available in literature among them the monotone iterative techniques is a powerful tool. The method of upper and lower solutions and the monotone iterative techniques for existence and approximation of solutions to initial and boundary value problems corresponding to ordinary/partial differential equations are well studied. However, for fractional differential equations(FDEs), the scheme is in its initial stage and only few articles can be found in the literature dealing with upper and lower solutions, we refer [17, 19–21, 25, 26, 30, 31] for some of the results. Such technique when use together with the method of upper and lower solutions fruitful results may be obtained. Zhang [29], investigated the existence and uniqueness of solutions for the following initial value problems of FDEs

$$\begin{cases} D_{0+}^{\alpha} u(t) = f(t, u(t)), & t \in (0, T], \\ t^{1-\alpha} u(t)|_{t=0} = u_0, \end{cases} \quad (1.1)$$

where  $0 < T < \infty$  and  $D^{\alpha}$  is the Riemann-liouville fractional derivative of order  $\alpha \in (0, 1)$ . G.Wang *et al* [25], developed sufficient conditions for the existence of upper and lower solutions by using monotone iterative techniques together with the method of upper and lower solutions for the following coupled system of FDEs

$$\begin{cases} D_{0+}^{\alpha} u(t) = f(t, u(t), v(t)), & t \in (0, T], \\ D_{0+}^{\alpha} v(t) = g(t, v(t), u(t)), & t \in (0, T], \\ t^{1-\alpha} u(t)|_{t=0} = x_0, & t^{1-\alpha} v(t)|_{t=0} = y_0, \end{cases} \quad (1.2)$$

where  $0 < T < \infty$ , and  $f, g : [0, T] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ ,  $x_0, y_0 \in \mathbb{R}$  and  $x_0 \leq y_0$ ,  $D^{\alpha}$  is the Riemann-liouville fractional derivative of order  $0 < \alpha \leq 1$ . X.Zhang *et al* [17], obtained sufficient conditions for the multiplicity of positive solutions for the following boundary value problems of FDEs of the form

$$\begin{cases} -D_t^{\alpha} x(t) = f(t, x(t), -D_t^{\beta} v(t)), & t \in (0, 1), \\ D_t^{\beta} x(0) = 0, & D_t^{\gamma} x(1) = \sum_{j=1}^{p-2} a_j D_t^{\gamma} x(\xi_j), \end{cases} \quad (1.3)$$



where  $1 < \alpha \leq 2, \alpha - \beta > 1, 0 < \beta \leq \gamma < 1, 0 < \xi_1 < \xi_2 < \dots < \xi_{p-2} < 1, a_j \in [0, +\infty)$  with  $\sum_{j=1}^{p-2} a_j \xi_j^{\alpha-\gamma-1} < 1$  and  $f : (0, 1) \times (0, +\infty) \times (-\infty, +\infty) \rightarrow [0, +\infty)$  is continuous and singular at  $t = 0, 1$  and  $D_t^\alpha$ , is the Riemann-Liouville fractional derivative. In recent years, the monotone iterative techniques together with the method of upper and lower solutions for coupled system of boundary value problems of fractional order differential-integral equations have attracted some attention. Existence of positive solutions for Neumann boundary value problems(NBVP) for fractional order differential equations has been rarely studied and very few articles are found in literature. Z.Hu *et al* [30], used coincidence degree theory to established sufficient conditions for the existence of positive solution for the following coupled systems of (NBVP)

$$\begin{cases} D_{0+}^\alpha u(t) = f(t, v(t), v'(t)), & t \in (0, 1), \\ D_{0+}^\beta v(t) = g(t, u(t), u'(t)), & t \in (0, 1), \\ u'(0) = u'(1) = 0, & v'(0) = v'(1) = 0, \end{cases} \tag{1.4}$$

where  $1 < \alpha, \beta \leq 2$  and  $f, g : [0, 1] \times \mathbb{R}^2 \rightarrow \mathbb{R}$  is continuous and  $D_{+0}^\alpha, D_{+0}^\beta$  is the standard Riemann-Liouville fractional derivative.

Motivated by the above mentioned work, we discuss the following coupled system of nonlinear fractional order differential-integral equations with mixed type Neumann boundary conditions of the form

$$\begin{cases} D^\alpha u(t) + f(t, v(t), I^{\beta-2}v(t), I^{\beta-1}v(t), I^\beta v(t)) = 0, & t \in I = [0, 1], \\ D^\beta v(t) + g(t, u(t), I^{\alpha-2}u(t), I^{\alpha-1}u(t), I^\alpha u(t)) = 0, & t \in I = [0, 1], \\ u(0) = 0, \text{ and } u'(0) = u'(1) = 0, & v(0) = 0, \text{ and } v'(0) = v'(1) = 0, \end{cases} \tag{1.5}$$

where  $2 < \alpha, \beta \leq 3$  and  $u, v \in C[0, 1]$  and the non-linear functions  $f, g : [0, 1] \times \mathbb{R}^4 \rightarrow \mathbb{R}$  satisfy the Caratheodory conditions. Also  $D^\alpha, D^\beta, I^\alpha, I^\beta$  denote fractional order derivative and fractional order integrations of order  $\alpha, \beta$  respectively in Riemann-Liouville sense. We apply the monotone iterative technique together with the method of upper and lower solutions to obtain sufficient conditions for existence and approximation of multiple solutions. An error estimation is also derive for consistency of the method. Further we provide an example to illustrated our results.

Rest of our paper is organize as in section 2, we provide some basic results and lemmas need for this study. Section 3 is devoted to the main result, while in sections 4, we provide an examples and a short conclusion.



## 2. PRELIMINARIES

We recall some basic definitions and known results from fractional calculus, functional analysis and measure theory, which can be found in [9, 12, 16, 18, 19, 26, 31].

**Definition 2.1.** The Riemann-Liouville fractional integral of order  $\alpha \in \mathbb{R}_+$  of a function  $h \in L([0, 1], \mathbb{R})$  is defined by

$$I_t^\alpha h(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds,$$

provided the integral converges.

**Definition 2.2.** The Riemann-Liouville fractional order derivative of a function  $h$  on the interval  $[0, 1]$  is defined by

$$D_t^\alpha h(t) = \frac{1}{\Gamma(n-\alpha)} \left( \frac{d}{dt} \right)^n \int_0^t (t-s)^{n-\alpha-1} h(s) ds,$$

where  $n = [\alpha] + 1$  and  $[\alpha]$  represents the integer part of  $\alpha$ .

**Definition 2.3.** Let  $X, Y$  be Banach spaces satisfying the property of partial order and  $C \subset X$ . An operator  $T : C \rightarrow Y$  is said to be increasing if for all  $x, y \in C$  with  $x \leq y$  implies  $Tx \leq Ty$ . Similarly  $T$  is decreasing if for all  $x, y \in C$  with  $x \geq y$  implies  $Tx \geq Ty$ .

**Definition 2.4.** A function  $\alpha^* \in C$  is said to a minimal solution of the operator equation  $(I - T)z = 0$  if  $(I - T)\alpha^* \leq 0$  and  $\beta^* \in C$  is said to an maximal solution of  $(I - T)z = 0$  if  $(I - T)\beta^* \geq 0$ .

**Definition 2.5.** [31]. A functions  $f(t, x, y) : [0, 1] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  is said to satisfies the Carathodeory conditions, if the following hold

- (i)  $f(t, x, y, u, v)$  is Lebesgue measurable with respect to  $t$  for each  $x, y, u, v \in \mathbb{R}$ ,
- (ii)  $f(t, x, y, u, v)$  is continuous for  $x, y, u, v$  for all most every  $t \in [0, 1]$ .

**Lemma 2.6.** [31]. Let  $X$  be a partial order Banach space and  $x_n, y_n \in X$  such that  $x_n \leq y_n, n \in \mathbb{Z}^+$ . If  $x_n \rightarrow x$  and  $y_n \rightarrow y$ , then  $x \leq y$ .

**Lemma 2.7.** [3]. For  $u \in C(0, 1) \cap L(0, 1)$ , the following result holds

$$I^\alpha D^\alpha u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + c_3 t^{\alpha-3} + \dots + c_n t^{\alpha-n},$$

where  $c_i \in \mathbb{R}, i = 1, 2, \dots, n$  are arbitrary constants.

Let  $X = C[0, 1]$ , then  $X$  is a Banach space endowed with the norm  $\|x\| = \max_{t \in [0, 1]} |x(t)|$ . A set  $C \subset X$  is called cone in  $X$  by defining a partial ordering



in  $X$  by  $x \leq y$  if and only if  $y - x \in C$ , such a Banach space is called a partial order Banach space. Let  $x_0, x_0^*$  be lower and upper solutions respectively of (1.5), with  $x_0 \leq x_0^*$ , we define a closed set  $\Omega = [x_0, x_0^*]$  required for further process. In this papers we need the following assumptions :

- (A<sub>1</sub>)  $f, g : [0, 1] \times \mathbb{R}^4 \rightarrow \mathbb{R}$  satisfies Carathodory conditions,
- (A<sub>2</sub>) For any  $x_i, y_i \in C$  with  $x_i \leq y_i (i = 1, 2, 3, 4)$ , there exist constants  $\mathcal{A}_i, \mathcal{B}_i (i = 1, 2, 3, 4) > 0$  such that
 
$$0 \leq f(t, y_1, y_2, y_3, y_4) - f(t, x_1, x_2, x_3, x_4) \leq \mathcal{A}_1(y_1 - x_1) + \mathcal{A}_2(y_2 - x_2) + \mathcal{A}_3(y_3 - x_3) + \mathcal{A}_4(y_4 - x_4), \quad t \in [0, 1],$$

$$0 \leq g(t, y_1, y_2, y_3, y_4) - g(t, x_1, x_2, x_3, x_4) \leq \mathcal{B}_1(y_1 - x_1) + \mathcal{B}_2(y_2 - x_2) + \mathcal{B}_3(y_3 - x_3) + \mathcal{B}_4(y_4 - x_4), \quad t \in [0, 1],$$
- (A<sub>3</sub>) Let  $(u_0, v_0)$  and  $(u_0^*, v_0^*) \in X \times X$  are lower and upper solution respectively of (1.5), then  $u_0 \leq u_0^*$  and  $v_0 \leq v_0^*$ .

### 3. MAIN RESULTS

**Theorem 3.1.** Under the assumption (A<sub>1</sub>) and Lemma (2.7), and for the integral representation of the system of Mixed Neumann type boundary value problems (1.5) is given by

$$\begin{cases} u(t) = \int_0^1 \mathcal{K}_1(t, s) f(s, v(s), I^{\beta-2}v(s), I^{\beta-1}v(s), I^\beta v(s)) ds, & t \in [0, 1], \\ v(t) = \int_0^1 \mathcal{K}_2(t, s) g(s, u(s), I^{\alpha-2}u(s), I^{\alpha-1}u(s), I^\alpha u(s)) ds, & t \in [0, 1], \end{cases} \quad (3.1)$$

where  $\mathcal{K}_i(t, s), i = 1, 2$  are given by

$$\mathcal{K}_1(t, s) = \frac{1}{\Gamma(\alpha)} \begin{cases} t^{\alpha-1}(1-s)^{\alpha-2} - (t-s)^{\alpha-1}, & 0 \leq s \leq t \leq 1, \\ t^{\alpha-1}(1-s)^{\alpha-2}, & 0 \leq t \leq s \leq 1, \end{cases} \quad (3.2)$$

$$\mathcal{K}_2(t, s) = \frac{1}{\Gamma(\beta)} \begin{cases} t^{\beta-1}(1-s)^{\beta-2} - (t-s)^{\beta-1}, & 0 \leq s \leq t \leq 1, \\ t^{\beta-1}(1-s)^{\beta-2}, & 0 \leq t \leq s \leq 1. \end{cases} \quad (3.3)$$

*Proof.* Applying  $I^\alpha$  on the first equation of the system (1.5), and using Lemma (2.7), we get

$$u(t) = -I^\alpha f(t, v(t), I^{\beta-2}v(t), I^{\beta-1}v(t), I^\beta v(t)) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + c_3 t^{\alpha-3}, \quad c_1, c_2, c_3 \in \mathbb{R}. \quad (3.4)$$



Since  $2 < \alpha \leq 3$ , one has  $n = 3$  due to Lemma (2.7) which implies that  $u(t)$  is singular at 0, also as  $u \in C[0, 1]$  as in [11, 12, 18] which implies that  $c_3 = 0$  in (3.4). Moreover by means of boundary conditions  $u'(0) = 0$ , and  $u'(1) = 0$  we get

$$c_2 = 0, \text{ and } c_1 = \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-2} f(s, v(s), I^{\beta-2}v(s), I^{\beta-1}v(s), I^\beta v(s)) ds.$$

Hence, (3.4) can be rewritten as

$$\begin{aligned} u(t) &= \frac{t^{\alpha-1}}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-2} f(s, v(s), I^{\beta-2}v(s), I^{\beta-1}v(s), I^\beta v(s)) ds \\ &\quad - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, v(s), I^{\beta-2}v(s), I^{\beta-1}v(s), I^\beta v(s)) ds \\ &= \int_0^1 \mathcal{K}_1(t, s) f(s, v(s), I^{\beta-2}v(s), I^{\beta-1}v(s), I^\beta v(s)) ds. \end{aligned}$$

Similarly, applying  $I^\beta$  on the second equation of the system (1.5) and Lemma (2.7), we get the second part of the system of integral equations (3.1)

$$\begin{aligned} v(t) &= \frac{t^{\beta-1}}{\Gamma(\beta)} \int_0^1 (1-s)^{\beta-2} g(s, u(s), I^{\alpha-2}u(s), I^{\alpha-1}u(s), I^\alpha u(s)) ds \\ &\quad - \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} g(s, u(s), I^{\alpha-2}u(s), I^{\alpha-1}u(s), I^\alpha u(s)) ds \\ &= \int_0^1 \mathcal{K}_2(t, s) g(s, u(s), I^{\alpha-2}u(s), I^{\alpha-1}u(s), I^\alpha u(s)) ds. \end{aligned}$$

□

**Lemma 3.2.** *The Green's functions  $\mathcal{K}_i(t, s)$  ( $i = 1, 2$ ), satisfy the following properties which are required in our results.*

$$(P_1) \quad \mathcal{K}_1(t, s) \geq 0, \mathcal{K}_2(t, s) \geq 0 \text{ for all } t, s \in [0, 1].$$

$$(P_2) \quad \int_0^1 \mathcal{K}_1(t, s) ds \leq \frac{1}{(\alpha-1)\Gamma(\alpha+1)}, \int_0^1 \mathcal{K}_2(t, s) ds \leq \frac{1}{(\beta-1)\Gamma(\beta+1)}, \text{ for all } t \in [0, 1].$$

*Proof.* ( $P_1$ ) : Since  $s, t \in [0, 1]$  and  $2 < \alpha \leq 3$ , so one has easily observe that  $t^{\alpha-1}(1-s)^{\alpha-2} \geq 0$  and also

$$t^{\alpha-1}(1-s)^{\alpha-2} \geq t^{\alpha-1}(1-s)^{\alpha-2}(1-s) = t^{\alpha-1}(1-s)^{\alpha-1}.$$

On the other hand,  $s \geq ts$  from which we have  $(t-s)^{\alpha-1} \leq (t-ts)^{\alpha-1} = t^{\alpha-1}(1-s)^{\alpha-1}$ . Thus we have

$$t^{\alpha-1}(1-s)^{\alpha-2} - (t-s)^{\alpha-1} \geq t^{\alpha-1}(1-s)^{\alpha-1} - t^{\alpha-1}(1-s)^{\alpha-1} = 0.$$



Hence  $\mathcal{K}_1(t, s) \geq 0, \forall s, t \in [0, 1]$ . Similarly we can show that  $\mathcal{K}_2(t, s) \geq 0, \forall s, t \in [0, 1]$ .

$$(P_2) : \int_0^1 \mathcal{K}_1(t, s) ds = \int_0^1 \frac{t^{\alpha-1}(1-s)^{\alpha-2}}{\Gamma(\alpha)} ds - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \leq \frac{1}{(\alpha-1)\Gamma(\alpha+1)}.$$

Similarly, we have  $\int_0^1 \mathcal{K}_2(t, s) ds \leq \frac{1}{(\beta-1)\Gamma(\beta+1)}$ . □

We write the system of integral equations (3.1) in the following equivalent form of integral equation

$$\begin{aligned} u(t) &= \int_0^1 \mathcal{K}_1(t, s) f(s, v(s), I^{\beta-2}v(s), I^{\beta-1}v(s), I^\beta v(s)) ds \\ &= \int_0^1 \mathcal{K}_1(t, s) f \left( s, \int_0^1 \mathcal{K}_2(s, x) g(x, u(x), I^{\alpha-2}u(x), I^{\alpha-1}u(x), I^\alpha u(x)) dx, \right. \\ &\quad \left. I^{\beta-2} \left[ \int_0^1 \mathcal{K}_2(s, x) g(s, u(x), I^{\alpha-2}u(x), I^{\alpha-1}u(x), I^\alpha u(x)) dx \right], \right. \\ &\quad \left. I^{\beta-1} \left[ \int_0^1 \mathcal{K}_2(s, x) g(s, u(x), I^{\alpha-2}u(x), I^{\alpha-1}u(x), I^\alpha u(x)) dx \right], \right. \\ &\quad \left. I^\beta \left[ \int_0^1 \mathcal{K}_2(s, x) g(s, u(x), I^{\alpha-2}u(x), I^{\alpha-1}u(x), I^\alpha u(x)) dx \right] \right) ds. \end{aligned} \tag{3.5}$$

and define an operator  $T : \Omega \rightarrow X$  by

$$\begin{aligned} Tu(t) &= \int_0^1 \mathcal{K}_1(t, s) f \left( s, \int_0^1 \mathcal{K}_2(s, x) g(x, u(x), I^{\alpha-2}u(x), I^{\alpha-1}u(x), I^\alpha u(x)) dx, \right. \\ &\quad \left. I^{\beta-2} \left[ \int_0^1 \mathcal{K}_2(s, x) g(s, u(x), I^{\alpha-2}u(x), I^{\alpha-1}u(x), I^\alpha u(x)) dx \right], \right. \\ &\quad \left. I^{\beta-1} \left[ \int_0^1 \mathcal{K}_2(s, x) g(s, u(x), I^{\alpha-2}u(x), I^{\alpha-1}u(x), I^\alpha u(x)) dx \right], \right. \\ &\quad \left. I^\beta \left[ \int_0^1 \mathcal{K}_2(s, x) g(s, u(x), I^{\alpha-2}u(x), I^{\alpha-1}u(x), I^\alpha u(x)) dx \right] \right) ds. \end{aligned} \tag{3.6}$$

Then, the integral equation (3.6) can be written as an operator equation

$$(I - Tu)(t) = 0, \quad t \in [0, 1], \tag{3.7}$$

and solutions of the operator equation (3.7) are the solutions of the integral equation (3.1), that is, fixed points of  $T$ , are the corresponding solutions of integral equations (3.1). In view of  $A_2$ , for  $u, v \in \Omega$  with  $u \leq v$ , we have  $I^{\alpha-i}u \leq I^{\alpha-i}v (i = 0, 1, 2)$ ,



as  $\mathcal{K}_2(t, s) \geq 0$ , we get

$$\begin{aligned}
Tu(t) &= \int_0^1 \mathcal{K}_1(t, s) f \left( s, \int_0^1 \mathcal{K}_2(s, x) g(s, u(x), I^{\alpha-2}u(x), I^{\alpha-1}u(x), I^\alpha u(x)) dx, \right. \\
&\quad I^{\beta-2} \left[ \int_0^1 \mathcal{K}_2(s, x) g(s, u(x), I^{\alpha-2}u(x), I^{\alpha-1}u(x), I^\alpha u(x)) dx \right], \\
&\quad I^{\beta-1} \left[ \int_0^1 \mathcal{K}_2(s, x) g(s, u(x), I^{\alpha-2}u(x), I^{\alpha-1}u(x), I^\alpha u(x)) dx \right], \\
&\quad \left. I^\beta \left[ \int_0^1 \mathcal{K}_2(s, x) g(s, u(x), I^{\alpha-2}u(x), I^{\alpha-1}u(x), I^\alpha u(x)) dx \right] \right) ds \\
&\leq \int_0^1 \mathcal{K}_1(t, s) f \left( s, \int_0^1 \mathcal{K}_2(s, x) g(s, v(x), I^{\alpha-2}v(x), I^{\alpha-1}v(x), I^\alpha v(x)) dx, \right. \\
&\quad I^{\beta-2} \left[ \int_0^1 \mathcal{K}_2(s, x) g(s, v(x), I^{\alpha-2}v(x), I^{\alpha-1}v(x), I^\alpha v(x)) dx \right], \\
&\quad I^{\beta-1} \left[ \int_0^1 \mathcal{K}_2(s, x) g(s, v(x), I^{\alpha-2}v(x), I^{\alpha-1}v(x), I^\alpha v(x)) dx \right], \\
&\quad \left. I^\beta \left[ \int_0^1 \mathcal{K}_2(s, x) g(s, v(x), I^{\alpha-2}v(x), I^{\alpha-1}v(x), I^\alpha v(x)) dx \right] \right) ds = Tv(t),
\end{aligned} \tag{3.8}$$

which implies that  $T$  is nondecreasing operator.

Use the notation

$$\begin{aligned}
\Delta &= \frac{\mathcal{A}_1 \mathcal{B}_1}{(\beta-1)\Gamma(\beta+1)} + \frac{\mathcal{A}_1 \mathcal{B}_2}{(\beta-1)\Gamma(\alpha-1)\Gamma(\beta+1)} + \frac{\mathcal{A}_1 \mathcal{B}_3}{(\beta-1)\Gamma\alpha\Gamma(\beta+1)} \\
&+ \frac{\mathcal{A}_1 \mathcal{B}_4}{\Gamma\alpha\Gamma\beta\Gamma(\beta+1)\Gamma(\alpha+1)} + \frac{\mathcal{A}_2 \mathcal{B}_1}{\Gamma\beta\Gamma(\beta+1)} + \frac{\mathcal{A}_2 \mathcal{B}_2}{\Gamma\beta\Gamma(\alpha-1)\Gamma(\beta+1)} \\
&+ \frac{\mathcal{A}_2 \mathcal{B}_3}{(\beta-1)\Gamma(\beta+1)} + \frac{\mathcal{A}_2 \mathcal{B}_4}{\Gamma\beta(\beta-1)\Gamma(\beta+1)} + \frac{\mathcal{A}_3 \mathcal{B}_1}{(\beta-1)\Gamma\beta\Gamma(\beta+1)} \\
&+ \frac{\mathcal{A}_3 \mathcal{B}_2}{(\beta-1)\Gamma\beta\Gamma(\beta+1)\Gamma(\alpha-1)\Gamma(\beta+1)} + \frac{\mathcal{A}_3 \mathcal{B}_3}{(\beta-1)\Gamma\beta\Gamma\alpha\Gamma(\beta+1)} \\
&+ \frac{\mathcal{A}_3 \mathcal{B}_4}{\Gamma\alpha\Gamma\beta\Gamma(\beta+1)\Gamma(\alpha+1)} + \frac{\mathcal{A}_4 \mathcal{B}_1}{(\beta-1)\Gamma^2(\beta+1)} \\
&+ \frac{\mathcal{A}_4 \mathcal{B}_2}{(\beta-1)\Gamma(\alpha)\Gamma^2(\beta+1)} + \frac{\mathcal{A}_4 \mathcal{B}_3}{(\beta-1)\Gamma(\alpha+1)\Gamma^2(\beta+1)} + \frac{\mathcal{A}_4 \mathcal{B}_4}{(\beta-1)\Gamma(\alpha+1)\Gamma^2(\beta+1)}.
\end{aligned}$$

**Theorem 3.3.** *Under the assumptions  $(A_1) - (A_3)$  and  $\Delta < 1$ , then there exists minimal and maximal solutions and a monotone sequence of solutions of linear problems converging uniformly to a solution of the nonlinear integral equation (3.6).*

**Proof:** In view of  $(A_1)$  and  $(A_2)$ , the operator  $T$  is continuous and nondecreasing. Assume that  $\mu, \nu \in X$  are lower and upper solutions of (3.7) such that  $\mu \leq \nu$  on  $[0, 1]$ . Choose  $u_0 = \mu$ , using the definition of lower and upper solutions and the nondecreasing property of  $T$ , we obtain

$$u_0 \leq Tu_0 \leq T\nu \leq \nu, \text{ that is, } u_0 \leq u_1 \leq \nu \text{ on } [0, 1],$$

where  $u_1$  is a solution of the linear problem  $u_1 = Tu_0$ . Again the nondecreasing property of  $T$  implies that

$$Tu_0 \leq Tu_1 \leq T\nu \leq \nu, \text{ that is, } u_1 \leq u_2 \leq \nu \text{ on } [0, 1],$$





where  $u_2$  is a solution of the linear problem  $u_2 = Tu_1$ . Continuing in the above fashion, we get a bounded monotone sequence  $\{u_n\}$  of solutions satisfies

$$u_0 \leq u_1 \leq u_2 \leq \dots u_{n-1} \leq u_n \leq \nu \text{ on } [0, 1], \tag{3.9}$$

where  $u_n$  is a solution of the linear problem  $u_n = Tu_{n-1}$ . The monotonicity and boundedness of the sequence implies the existence of  $u \in \Omega$  such that  $u_n \rightarrow u$  as  $n \rightarrow \infty$ . Hence, passing to the limit  $n \rightarrow \infty$ , the equation  $u_n = Tu_{n-1}$  implies that  $u = Tu$ , that is,  $u$  is a solution of the integral equation (3.6). Moreover from  $(A_3)$  and for any  $u, v \in \Omega$  with  $I^{\alpha-i}u \leq I^{\beta-i}v, i = 0, 1, 2$ , and

$$g(t, u, I^{\alpha-2}u(t), I^{\alpha-1}u(t), I^\alpha u(t)) \leq g(t, v(t), I^{\alpha-2}v(t), I^{\alpha-1}v(t), I^\alpha v(t)), \mathcal{K}_2(t, s) \geq 0,$$

we obtain

$$\begin{aligned} & \left\| f \left( t, \int_0^1 \mathcal{K}_2(t, x)g(x, v(x), I^{\alpha-1}v(x), I^{\alpha-2}v(x), I^\alpha v(x))dx, \right. \right. \\ & I^{\beta-2} \left[ \int_0^1 \mathcal{K}_2(t, x)g(x, v(x), I^{\alpha-1}v(x), I^{\alpha-2}v(x), I^\alpha v(x)) \right], \\ & I^{\beta-1} \left[ \int_0^1 \mathcal{K}_2(t, x)g(x, v(x), I^{\alpha-1}v(x), I^{\alpha-2}v(x), I^\alpha v(x))dx \right], \\ & \left. I^\beta \left[ \int_0^1 \mathcal{K}_2(t, x)g(x, v(x), I^{\alpha-1}v(x), I^{\alpha-2}v(x), I^\alpha v(x))dx \right] \right) \\ & - f \left( t, \int_0^1 \mathcal{K}_2(t, x)g(x, u(x), I^{\alpha-1}u(x), I^{\alpha-2}u(x), I^\alpha u(x))dx, \right. \\ & I^{\beta-2} \left[ \int_0^1 \mathcal{K}_2(t, x)g(x, u(x), I^{\alpha-1}u(x), I^{\alpha-2}u(x), I^\alpha u(x))dx \right], \\ & I^{\beta-1} \left[ \int_0^1 \mathcal{K}_2(t, x)g(x, u(x), I^{\alpha-1}u(x), I^{\alpha-2}u(x), I^\alpha u(x))dx \right], \\ & \left. \left. I^\beta \left[ \int_0^1 \mathcal{K}_2(t, x)g(x, u(x), I^{\alpha-1}u(x), I^{\alpha-2}u(x), I^\alpha u(x))dx \right] \right) \right\| \end{aligned} \tag{3.10}$$

$$\|Tv - Tu\| \leq \Delta \|v - u\| \tag{3.11}$$

Now from (3.9) and (3.11) we have

$$\begin{aligned} \|u_2 - u_1\| &= \|Tu_1 - Tu_0\| \leq \Delta \|u_1 - u_0\|, \\ \|u_3 - u_2\| &= \|Tu_2 - Tu_1\| \leq \Delta^2 \|u_1 - u_0\|, \\ \|u_4 - u_3\| &= \|Tu_3 - Tu_2\| \leq \Delta^3 \|u_1 - u_0\|, \\ &\dots \dots \dots \end{aligned}$$

From these results we have  $\|u_{n+1} - u_n\| \leq \Delta^n \|u_1 - u_0\|$ . So for any  $m, n \in \mathbb{Z}^+$ , we get

$$\begin{aligned} \|u_{m+n} - u_n\| &\leq \|u_{m+n} - u_{m+n-1}\| + \|u_{m+n-1} - u_{m+n-2}\| + \dots + \|u_{n+1} - u_n\| \\ &\leq \Delta^n \frac{1 - \Delta^m}{1 - \Delta} \|u_1 - u_0\|. \end{aligned}$$



(3.12)

For  $0 < \Delta < 1$ , we have  $\|u_{m+n} - u_n\| \rightarrow 0$ , as  $n \rightarrow \infty$ , therefore  $u_n$  is a cauchy sequence in  $\Omega$ . Let  $u(t) = \lim_{n \rightarrow \infty} u_n(t) \Rightarrow Tu = u$ . Thus (1.5) has a pair of solution  $(u, v)$ . Further if  $m \rightarrow \infty$  in (3.12) we have an error estimate for lower solutions as  $\|u - u_n\| \leq \frac{\Delta^n}{1-\Delta} \|u_1 - u_0\|$ . Define the error  $e_n = u - u_n$ , since  $u_{n-1} \leq u_n \Rightarrow u - u_{n-1} \geq u - u_n$ , which gives that  $e_{n-1} \geq e_n$ , hence  $e_n$  is monotonically decreasing sequence which converges to its lower bound i.e

$$\lim_{n \rightarrow \infty} e_n = \lim_{n \rightarrow \infty} (u - u_n) = 0 \Rightarrow \lim_{n \rightarrow \infty} u_n = u.$$

Thus the error estimation for lower solution is

$$\|u - u_n\| \leq \frac{\Delta^n}{1-\Delta} \|u_1 - u_0\|.$$

Similarly for initial iteration of upper solutions  $\nu$  using  $u_0^* = \nu$  for (3.6) we get

$$u_0 \leq u_1 \leq u_2 \leq \dots \leq u_n \leq \dots \leq u_n^* \leq \dots \leq u_2^* \leq u_1^* \leq u_0^* \text{ on } [0, 1].$$

Similarly the error estimation for upper solution is given by

$$\|u_n^* - u^*\| \leq \frac{\Delta^n}{1-\Delta} \|u_0^* - u_1^*\|.$$

Further, we prove the existence of maximal and minimal solution of (1.5). Let  $(u_0, v_0)$  and  $(u_0^*, v_0^*)$  be minimal and maximal solutions of (1.5), for any  $w(t) \in \Omega$  with  $Tw = w$ , we obtain  $u_n \leq w \leq u_n^*$ , as  $T$  is increasing operator so we have  $Tu_n \leq Tw \leq Tu_n^*$  as  $n \rightarrow \infty$  we get  $u(t) \leq w(t) \leq u^*(t)$ . Which implies that  $u, u^*$  are minimal and maximal fixed point of operator  $T$  respectively. So there exist minimal and maximal solutions of (1.5) in the form  $(u_0, v_0)$  and  $(u_0^*, v_0^*)$  respectively. Hence proof is completed.  $\square$

**Theorem 3.4.** *Under the assumptions  $(A_1) - (A_3)$  and  $\Delta < 1$ , then the system of BVP (1.5) has a unique maximal and minimal solution .*

*Proof.* Let  $x_0, y_0 \in X$  be lower and upper solution of operator equation  $Tu = u$  respectively such that  $x_0 \leq Tx_0 \leq Ty_0 \leq y_0, t \in [0, 1]$ . Therefore  $x_n \rightarrow x^*, y_n \rightarrow y^*, n \rightarrow \infty$ , also we have  $Tx^* = x^*, Ty^* = y^*$ .

To prove uniqueness of solution i.e  $x^* = u^*$ , then as  $u_0 \leq x^*$  and due to increasing property of  $T$  on  $\Omega$ . We have  $u_n = T^n u_0 \leq T^n x^*$ , for each  $n \in Z^+$ . Thus  $u_0 \leq u_1 \leq u_2 \leq \dots \leq x^*$ . By mathematical inductions and using  $\Delta < 1$  we have  $\|x^* - u^*\| = \|T^n x^* - T^n u_0\| \leq \Delta^n \|x^* - u_0\| \rightarrow 0, (n \rightarrow \infty) \Rightarrow u^* = x^*$ , similarly  $v^* = y^*$ . Thus uniqueness of minimal and maximal solution has been followed for (1.5).  $\square$



4. EXAMPLE

Consider the following coupled system of boundary values problem

$$\left\{ \begin{array}{l} D^{2.5}u(t) + \frac{t^2}{16}v(t) \sin(t) + \frac{t^3}{16}I^{0.5}v(t) + \left(\frac{1-t}{4}\right)^2 I^{1.5}v(t) \\ + \left(\frac{1-t}{4}\right)^3 I^{2.5}v(t) = 0, \quad t \in [0, 1], \\ D^{2.5}v(t) + \left(\frac{t}{3}u(t) \cos(t)\right)^3 + \frac{t^3}{16}I^{0.5}u(t) + \left(\frac{(1-t)^2}{25}\right)^2 I^{1.5}u(t) \\ + \left(\frac{(1-t)^4}{256}\right)^3 I^{2.5}u(t) = 0, \quad t \in [0, 1], \\ u(0) = 0, \text{ and } u'(0) = u'(1) = 0, \quad v(0) = 0 \text{ and } v'(0) = v'(1) = 0. \end{array} \right. \quad (4.1)$$

**Solution:** Here

$$\begin{aligned} f(t, v, I^{\beta-2}v(t), I^{\beta-1}v(t), I^\beta v(t)) &= \frac{t^2}{16}v(t) \sin(t) + \frac{t^3}{16}I^{0.5}v(t) \\ &+ \left(\frac{(1-t)^2}{16}\right)^2 I^{1.5}v(t) + \left(\frac{1-t}{4}\right)^3 I^{2.5}v(t), \\ g(t, u, I^{\alpha-2}u(t), I^{\alpha-1}u(t), I^\alpha u(t)) &= \frac{t^3}{27}u(t) \cos(t) + \frac{t^3}{16}I^{0.5}u(t) \\ &+ \left(\frac{(1-t)^2}{25}\right)^2 I^{1.5}u(t) + \left(\frac{(1-t)^4}{256}\right)^3 I^{2.5}u(t), \end{aligned}$$

$\alpha = 2.5, \beta = 2.5$ . For any  $u(t) \leq v(t)$  we have

$$0 \leq (f(t, v, I^{0.5}v(t), I^{1.5}v(t), I^{2.5}v(t)) - f(t, u, I^{0.5}u(t), I^{1.5}u(t), I^{2.5}u(t))),$$

and

$$0 \leq (g(t, v, I^{0.5}v(t), I^{1.5}v(t), I^{2.5}v(t)) - g(t, u, I^{0.5}u(t), I^{1.5}u(t), I^{2.5}u(t))),$$

it follows that

$$\mathcal{A}_1 = \frac{1}{16}, \mathcal{A}_2 = \frac{1}{16}, \mathcal{A}_3 = \frac{1}{16}, \mathcal{A}_4 = \frac{1}{64}, \alpha = 2.5,$$

and

$$\mathcal{B}_1 = \frac{1}{27}, \mathcal{B}_2 = \frac{1}{64}, \mathcal{B}_3 = \frac{1}{25}, \mathcal{B}_4 = \frac{1}{16}, \beta = 2.5.$$

Let us take  $(-1, -1)$  and  $(1, 1)$  as initial iteration for lower and upper solutions respectively. Then one can easily calculate that  $\Delta = .02509 < 1$ , thus uniqueness of maximal and minimal solutions has been followed. Clearly  $(0, 0)$  is the unique solution. To obtain error estimations we give iterative sequences by taking  $n$



is large enough. Let us take  $n = 7$ , then

$$u(t) = u_7(t),$$

$$v(t) = \int_0^1 \mathcal{K}_2(t, s)$$

$$\left[ \left( \frac{s}{3} u_7(s) \cos(s) \right)^3 + \frac{s^3}{16} I^{0.5} u_7(s) + \left( \frac{(1-s)^2}{25} \right)^2 I^{1.5} u_7(s) + \left( \frac{(1-s)^4}{256} \right)^3 I^{2.5} u_7(s) \right] ds$$

$$u^*(t) = u_7^*(t),$$

$$v^*(t) = \int_0^1 \mathcal{K}_2(t, s)$$

$$\left[ \left( \frac{s}{3} u_7^*(s) \cos(s) \right)^3 + \frac{s^3}{16} I^{0.5} u_7^*(s) + \left( \frac{(1-s)^2}{25} \right)^2 I^{1.5} u_7^*(s) + \left( \frac{(1-s)^4}{256} \right)^3 I^{2.5} u_7^*(s) \right] ds$$

$$e_7 = \|u - u_7\| \leq \frac{\Delta^7}{1 - \Delta} \|u_1(t) - u_0(t)\| \leq \frac{\Delta^7}{1 - \Delta} \max_{t \in [0,1]} \|u_1(t) + 1\| \approx 2.288 \times 10^{-12},$$

and similarly for upper solutions error estimation is

$$e_7 = \|u - u_7^*\| \leq \frac{\Delta^7}{1 - \Delta} \|u_1^*(t) - u_0^*(t)\| \leq \frac{\Delta^7}{1 - \Delta} \max_{t \in [0,1]} \|u_1^*(t) - 1\| \approx 1.88 \times 10^{-12}.$$

## 5. CONCLUSION

In this article, we have successfully developed an iterative scheme for maximal and minimal solutions to a coupled system of highly nonlinear fractional order differential equations. Where the nonlinearities explicitly depend on the terms involving fractional order integral which are very rarely studied. The concerned procedure has been enriched by developing conditions for error estimation. More over with the help of an example, we have verified the results been established in this article.

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