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Solution of bang - bang optimal control problems by using bezier polynomials

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Abstract In this paper, a new numerical method is presented for solving the optimal control problems of bang-bang type with free or fixed terminal time. The method is based on Bezier polynomials which are presented in any interval as $[t_0, t_f]$. The problems are reduced to a constrained problems which can be solved by using Lagrangian method. The constraints of these problems are terminal state and conditions. Illustrative examples are included to demonstrate the validity and applicability of the method.

Keywords. Optimal control, Bang-Bang control, Minimum-time, Bezier polynomialsfamily, Best approximation.

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1. Introduction

Bang-bang control switching between upper and lower bounds, is the optimal strategy for solving a wide variety of control problems. The control of a dynamical system has lower and upper bounds and the system model is linear in the input and nonsingular. Bang-bang control is often the appropriate choice because of the nature of the actuator of the physical system.

Switching from one mode of the system to another one can be modelled by a bang-bang type control. Bang-bang type controls arise in well-known application areas such as robotics, rocket flights , cranes, and also in applied physics [1, 2]. There are a large number of research papers that employ this method to solve optimal control problems (see for example [5, 6, 10, 11, 12, 13, 14, 15, 16, 25, 26, 27] and the references

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therein).

In this paper, a numerical method is presented to solve a class of bang-bang constrained optimal control problems, where the focus is on time-optimal control. First the optimal control problem is reduced to constrianed problems with equal constraint defined by the terminal state in time space. Also a computational method is presented for solving linear constrained quadratic optimal control problems by using Bezier polynomials. The method is based on approximating the state and the control variables with Bezier polynomials. Our method consists of reducing the optimal control problem to an NLP one by first expanding the state rate $\dot{x}(t)$ as a Bezier polynomial expansion with unknown coefficients and the control $u(t)$ as the bang-bang control. The operational matrix of differentiation D_Φ is obtained in order to approximate the differential part of the problem. The paper is organized as follows:

In Section 2, we describe the basic formulation of the Bezier functions required for our subsequent development. Section 3 is devoted to the formulation of bang-bang optimal control problems. Section 4 summarizes the application of this method to the optimal control problems, and in Section 5 we report our numerical result along with showing the accuracy of the proposed method.

2. Some properties of bernstein and bezier polynomials on $[t_0, t_f]$

The Bernstein basis polynomials of degree n on $[t_0, t_f]$ are defined as

$$B_{i,n}(t) = \binom{n}{i}(t-t_0)^i(t_f-t)^{n-i}, \;\; i \in [0,n], \tag{2.1}$$

where i is integer number and the binomial coefficients are given by

$$\binom{n}{i} = \begin{cases} \frac{n!}{i!(n-i)!}, & i \in [0,n], \\ 0, & elsewhere. \end{cases}$$

Some properties of these polynomials are

(*i*) $B_{i,n}(t_0) = \delta_{i,0}(t_f-t_0)^n$ and $B_{i,n}(t_f) = \delta_{i,n}(t_f-t_0)^n$, where δ is the Kronecker delta function.
(*ii*) $B_{i,n}(t)$ has two roots, with multiplicities i at $t=t_0$ and $n-i$ at $t=t_f$.
(*iii*) $B_{i,n}(t) \geq 0$ for $t \in [t_0, t_f]$ and $B_{i,n}(t_f-t) = B_{n-i,n}(t-t_0)$.
(*iv*) The Bernstein polynomials form a partition of unity i.e.$\sum_{i=0}^{n} B_{i,n}(t) = (t_f-t_0)^n$.
(v) Recursion: $B_{i,n}(t) = (t_f-t)B_{i,n-1}(t) + (t-t_0)B_{i-1,n-1}(t)$.
(vi) Derivative: $\frac{dB_{i,n}(t)}{dt} = \frac{1}{t_f-t_0}[(n+1-i)B_{i-1,n}(t)+(2i-n)B_{i,n}(t)-(i+1)B_{i+1,n}(t)]$.

2.1. **Definition of bezier polynomials on $[\mathbf{t_0}, \mathbf{t_f}]$.** We will express Bezier (polynomials) curves in terms of Bernstein polynomials, defined explicitly by

$$p_n(t) = \sum_{i=0}^{n} c_i B_{i,n}(t), \;\; t \in [t_0, t_f], \tag{2.2}$$

where $c_i = c[t_0^{<n-i>}, t_f^{<i>}]$ named control points or Bezier pionts and $t_0^{<n-i>}$ means that t_0 appears $n-i$ times. For example, $c[t_0^{<3>}, t_f^{<0>}] = c[t_0, t_0, t_0]$. Some properties of Bezier polynomials on $[t_0, t_f]$ are
(i) Symmetry: $\sum_{i=0}^{n} c_i B_{i,n}(t-t_0) = \sum_{i=0}^{n} c_{n-i} B_{i,n}(t_f - t)$.
(ii) Linear precision: $\frac{1}{(t_f-t_0)^{n-1}} \sum_{i=0}^{n} \frac{i}{n} B_{i,n}(t) = t - t_0$.

2.2. **The operational matrix of the bezier polynomials.** Consider

$$\Phi_n(t) = [B_{0,n}(t), B_{1,n}(t), ..., B_{n,n}(t)]^T, \ t \in [t_0, t_f], \tag{2.3}$$

where T denotes transposition.

2.3. **The operational matrix of derivative.** The differentiation of vector $\Phi_n(t)$ can be expressed as

$$\Phi_n'(t) = D_\Phi \Phi_n(t), \tag{2.4}$$

where D_Φ is the $(n+1)(n+1)$ operational matrix of derivative for the Bezier polynomials $B_{i,n}(t)$ which $t \in [t_0, t_f]$ and satisfies in

$$D_\Phi = \frac{1}{t_f - t_0} D_\phi, \tag{2.5}$$

where D_ϕ is the $(n+1)(n+1)$ operational matrix of derivative for Bezier polynomials $B_{i,n}(u)$ which $u \in [0,1]$ and $i = 0, ..., n$.

2.4. **Function approximation.** Suppose that $H = L^2[t_0, t_f]$ is a Hilbert space with the inner product defined as $< f.g > = \int_{t_0}^{t_f} f(t)g(t)dt$ and because the set $\{B_{0,n}(t), B_{1,n}(t), ..., B_{n,n}(t)\}$ is a complete basis in Hilbert space H then, any polynomial $B(t)$ of degree n can be expanded in terms of $B_{i,n}(t), i = 0, \ldots, n$ as follows

$$B(t) = \sum_{i=0}^{n} c_i B_{i,n}(t). \tag{2.6}$$

Also $\Phi_n(t) \subset H$ is the set of Bezier polynomials of degree n. Let S_n=span $\{\Phi_n(t)\}$ and f be an arbitrary element in H. Since S_n is a finite dimensional and closed subspace, therefore S_n is a complete subset of H. So, f has the unique best approximation out of S_n such as $S_0 \notin S_n$. So, there exist the unique coefficients c_i, i=0,..., n such that [16] any function $f \in H$ can be approximated in terms of Bezier polynomials as

$$f(t) \simeq S_0 = \sum_{i=0}^{n} c_i B_{i,n}(t) = C^T \Phi_n(t), \tag{2.7}$$

where $C = [c_0, \ldots, c_n]^T$ can be obtained as

$$\begin{aligned} C^T < \Phi_n.\Phi_n > = < f.\Phi_n > &= \int_{t_0}^{t_f} f(t)\Phi_n(t)dt \\ &= [< f.B_{0,n} >, \ldots, < f.B_{n,n} >]. \end{aligned} \tag{2.8}$$

Let $R =< \Phi_n.\Phi_n >$ be $(n+1)\times(n+1)$ matrix called the dual matrix of $\Phi_n(t)$, which can be determined by

$$R_{i+1,j+1} =< B_{i,n}.B_{j,n} > = \int_{t_0}^{t_f} B_{i,n}(t)B_{j,n}(t)dt$$
$$= (t_f - t_0)^{2n+1} \frac{\binom{n}{i}\binom{n}{j}}{(2n+1)\binom{2n}{i+j}}, \tag{2.9}$$

where $i, j = 0, \ldots, n$. We explain and prove the folowing lemma from [18].

Lemma 2.1. *Suppose that the function $f : [t_0, t_f] \to R$ be $n+1$ times continuously differentiable (i.e.$f \in C^{n+1}[t_0, t_f]$), and S_n=span$\{\Phi_n(t)\}$. If $C^T B$ is the best approximation of f out of S_n, then*

$$|f - C^T B|_{L^2[t_0,t_f]} \leq \frac{\hat{K}}{(n+1)!}\sqrt{\frac{t_f^{2m+3} - t_0^{2n+3}}{2n+3}}, \tag{2.10}$$

where $\hat{K}$=max$|f^{(n+1)}(t)|$, $t \in [t_0, t_f]$.

Proof. We know that $\{1, x, x^2, ..., x^n\}$ is a basis for polynomials space of degree $\leq n$. Therefore we define $y_1(x) = f(t_0) + xf'(t_0) + \frac{x^2}{2!}f''(t_0) + ... + \frac{x^n}{n!}f^{(n)}(t_0)$. By the Taylor expansion we have

$$|f(x) - y_1(x)| = |f^{(n+1)}(\xi_x)\frac{x^{n+1}}{(n+1)!}|, \tag{2.11}$$

where $\xi_x \in (t_0, t_f)$. Since $c^T B$ is the best approximation of f out of S_n. Then $y_1 \in S_n$ and from Eq. (2.11) we have

$$\begin{aligned}
\|f - c^T B\|^2_{L^2[t_0,t_f]} &\leq \|f - y_1\|^2_{L^2[t_0,t_f]} \\
&= \int_{t_0}^{t_f} |f(x) - y_1(x)|^2 dx \\
&= \int_{t_0}^{t_f} |f^{(n+1)}(\xi_x)|^2 (\frac{x^{n+1}}{(n+1)!})^2 dx \\
&\leq (\frac{\hat{K}}{(n+1)!})^2 \int_{t_0}^{t_f} x^{2n+2} dx \\
&= (\frac{\hat{K}}{(n+1)!})^2 (\frac{t_f^{2n+3} - t_0^{2n+3}}{2n+3}) \\
&= (\frac{\hat{K}}{(n+1)!})^2 (\frac{t_f^{2n+3}(1 - (\frac{t_0}{t_f})^{2n+3})}{2n+3}) \\
&\cong (\frac{\hat{K} t_f^n}{(n+1)!})^2 (\frac{t_f^3}{2n+3}).
\end{aligned}$$

Then by taking square roots, the proof is complete. We can rewrite Eq. (2.10) as following

$$
\begin{aligned}
|f - C^T B|_{L^2[t_0,t_f]} &\leq \frac{\hat{K}}{(n+1)!}\sqrt{\frac{t_f^{2m+3} - t_0^{2n+3}}{2n+3}} \\
&\cong \frac{\hat{K}}{(n+1)!}\sqrt{\left(\frac{t_f^{2n+3}(1-(\frac{t_0}{t_f})^{2n+3})}{2n+3}\right)} \\
&\cong \frac{\hat{K}t_f^n}{(n+1)!}\sqrt{\frac{t_f^3}{2n+3}}. \qquad (2.12)
\end{aligned}
$$

This Lemma shows that the error vanishes as $n \to \infty$.

3. Optimal control problems of bang-bang type

Optimal control problems of bang-bang type are problems where control function includes an uncontinuse point and the control on the environment of that point only has minimize and maximize amount. Now we considere optimal control problems of bang-bang type with normalized constraint $| u | \leq 1$. If a switching point exists, then control sequence will be as $\{-1, 1\}$ or $\{1, -1\}$. For better perception, considere equations system as following

$$\dot{x}(t) = Ax(t) + Bu(t). \qquad (3.1)$$

If we want this system arrive from primary state x_0 to terminal state x_1 in the least time, in a case that trajectory state and control are optimal. Hamiltonian function is obtained for (4.1) using pontryagin maximum principle on two domain space as following

$$H = -1 + \lambda_1 \dot{x_1} + \lambda_2 \dot{x_2} + (b_1\lambda_1 + b_2\lambda_2)u, \qquad (3.2)$$

where $B = [b_1, b_2]^T$. Because H is linear base u, thus for making maximum of H , we should have $u = 1$ or $u = -1$. Because chosing these amounts are dependant on the sign of fourth coefficient on H , thus only controls that can led to minimizing transfering time are as following

$$u^* = sgn(b_1\lambda_1 + b_2\lambda_2). \qquad (3.3)$$

This control is scrap constant that its discontinuity points will be on the zero places of following function

$$S = b_1\lambda_1 + b_2\lambda_2, \qquad (3.4)$$

namely control switching from 1 to -1 and vice versa occurs when we have $S = 0$ and as a result the function S is called switching function. Lemma is speech of this truth.

Lemma 3.1. *a: If eigenvalues of A are real, then switching function has at most one root.*
b: In this state a sequence of controls which are optimal are as following:

$$\{1\}, \{-1\}, \{-1, 1\}, \{1, -1\}.$$

singular controls. Becuase in some problems, switching function S is zero for all amounts of t that $t_1 \leq t \leq t_2$, thus control amounts u is not obtinated from $u = sgn(S)$ that in this case control function is called singular and becuase singular controls always aren't optimal, thus we search for unsingular controls that are exclusively optimal .For example in the fuel optimal control problems the fuel is used for guidance of vehicles from primal piont to final piont, that we want this used fuel be arrived to the least amount. It seems logical that consumable fuel amount on the base of time, will be multiplety of bigness forse of conveying. Therefore consumable fuel will equal with $\mid u \mid$. In this case only unsingular controls are as following

$$\{-1, 0, 1\}, \{0, 1\}, \{1\}, \{1, 0, -1\}, \{0, -1\}, \{-1\}.$$

4. Problem statement

Consider the nonlinear system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \tag{4.1}$$

$$x(t_0) = x^0, \;\; x(t_f) = x^1, \tag{4.2}$$

$$u(t) \in [a, b], \tag{4.3}$$

where $t \in [t_0, t_f]$, t_f is the terminal time free or fixed, $A(t) = (a_{i,j}(t))_{n\times n}$, $B(t) = (b_{i,j}(t))_{m\times m}$ are matrices, $x(t)$ is $n \times 1$ state vector, the control $u : [t_0, t_f] \to [a, b]$, $\dot{x}(t) : R^n \times [a, b] \to R^n$ is smooth in x except at the time points where the control u switches between a and b. The problem is to find the switching points $u(t)$ and the corresponding state trajectory $x(t)$ satisfying Eqs. (4.1), (4.2) and (4.3) while minimizing (or maximizing) the quadratic performance index

$$Z = \frac{1}{2}x^T(t_f)Gx(t_f) + \frac{1}{2}\int_{t_0}^{t_f} (x^T(t)Q(t)x(t) + u^T(t)R(t)u(t))dt, \tag{4.4}$$

where $G(t) = (g_{i,j}(t))_{n\times n}$, $Q(t) = (q_{i,j}(t))_{n\times n}$ are symmetric positive semi-definite matrices and and $R(t) = (r_{i,j}(t))_{m\times m}$ is a symmetric positive definite matrix.

4.1. Variational problems.

Consider the variational problem

$$Z(x(t)) = \int_{t_0}^{t_f} F(t, x(t), \dot{x}(t), \ldots, x^{(n)}(t))dt, \tag{4.5}$$

with the boundary conditions as

$$x(t_0) = a_0, \;\; \dot{x}(t_0) = a_1, \; \ldots, \;\; x^{(n-1)}(t_0) = a_{n-1}, \tag{4.6}$$

$$x(t_f) = b_0, \;\; \dot{x}(t_f) = b_1, \; \ldots, \;\; x^{(n-1)}(t_f) = b_{n-1}, \tag{4.7}$$

where $x(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^T$. The problem is to find the extremum of (4.5), subject to boundary conditions (4.6) and (4.7). The method consists of reducing the variational problem into a set of algebraic equations by first expanding $x(t)$ in terms of Bezier polynomials with unknown coefficients.

5. The proposed method

Let

$$t \in [t_j, t_{j+1}], \tag{5.1}$$
$$x_i^j(t) \simeq \Phi_n^j(t)^T X_i^j, \tag{5.2}$$
$$u^*(t) = u_j, \tag{5.3}$$

where $X_i^j, i = 1, \ldots, n$, are state coefficient vectors on $[t_j, t_{j+1}]$ trajectory. Then by using Eq. (2.4) we get

$$\dot{x}_i^j(t) \simeq [D_\Phi^j \Phi_n^j(t)]^T X^i. \tag{5.4}$$

By using Eqs. (5.1) and (5.3) we will have

$$x^j(t) \simeq [\Phi_n^j(t)^T X^j]^T = [\sum_{r=0}^{n} B_{r,n}^j(t) X_{1r}^j, \ldots, \sum_{r=0}^{n} B_{r,n}^j(t) X_{nr}^j]^T, \tag{5.5}$$

where $X^j = (X_{ir}^j)_{n\times(n+1)}$ is the state coefficient matrix. The boundary conditionsin Eq. (4.2) can be rewritten as

$$x^1(t_0) = x_0^1 = d_0^1 \otimes E \Phi_n^1(t), \tag{5.6}$$
$$x^m(t_f) = x_1^m = d_1^m \otimes E \Phi_n^1(t), \tag{5.7}$$
$$D_\Phi^j = \frac{1}{t_{j+1} - t_j} D_\phi, \tag{5.8}$$

where $t_m = t_f$, d_0^1 and d_1^m are $n \times 1$ constant vectors, $E = [1, \ldots, 1]$ is $1 \times (n+1)$ constant vector, and the symbol $\otimes$ denotes the Kronecker product [33]. If $x^1(t_0)$ or $x^m(t_f)$ are unknown in Eq. (4.2), then we put

$$x^1(t_0) \simeq [\Phi_n^1(t_0)^T X^1]^T = [\sum_{r=0}^{n} B_{r,n}^1(t_0) X_{1r}^1, \ldots, \sum_{r=0}^{n} B_{r,n}^1(t_0) X_{nr}^1]^T, \tag{5.9}$$
$$x^m(t_f) \simeq [\Phi_n^T(t_f) X^m]^T = [\sum_{r=0}^{n} B_{r,n}^m(t_f) X_{1r}^m, \ldots, \sum_{r=0}^{n} B_{r,n}^m(t_f) X_{nr}^m]^T. \tag{5.10}$$

5.1. **Performance index approximation.** By substituting Eqs. (5.6), (5.7) and (5.8) into Eq (4.5) we get

$$\begin{aligned} \min(\max) Z_j = {} & \frac{1}{2} x_1^{jT} G_1^j x_1^j + \frac{1}{2} X^{jT} [\int_{t_j}^{t_{j+1}} {\Phi_n^j}^T(t) Q^j(t) \Phi_n^j(t) dt] X^j \\ & + \frac{u_j^2}{2} [\int_{t_j}^{t_{j+1}} R^j(t) dt]. \end{aligned} \tag{5.11}$$

Let

$$P_x^j = \int_{t_j}^{t_{j+1}} \Phi_n^{jT}(t) Q^j(t) \Phi_n^j(t) dt, \quad \text{and} \quad P_u^j = \int_{t_j}^{t_{j+1}} R^j(t) dt. \tag{5.12}$$

By substituting Eq (5.12) in Eq (5.11) we get

$$Z_j = Z[X^j, u_j] = \frac{1}{2} X^{jT} (\hat{P}^j + P_x^j) X^j + \frac{1}{2} u_j^2 P_u^j, \tag{5.13}$$

where $\hat{P}^j = (\Phi_n^{jT} G_1^j \Phi_n^j)(t_{j+1})$. and $x_0^j = x^j(t_j), x_1^j = x^j(t_{j+1})$.

The boundary conditions in Eq. (4.2) can be expressed as

$$q_0^1 = x^1(t_0) - x_0^1, \qquad q_0^1 = (q_{0i}^1),\ i = 1, \dots, n, \tag{5.14}$$

$$q_1^m = x^m(t_f) - x_0^m, \quad q_1^m = (q_{0i}^m),\ i = 1, \dots, n. \tag{5.15}$$

We now find the extremum of Eq. (5.12) subject to Eqs. (5.13) and (5.14) by using the Lagrange multiplier method. Let

$$Z[X^j, u_j, \lambda_0^j, \lambda_1^j] = Z[X^j, u_j] + \lambda_0^j Q_0^j + \lambda_1^j Q_1^j, \tag{5.16}$$

where $Q_0^j = (q_{0i}^j), i = 1, \dots, n$ and $Q_1^j = (q_{1i}^j),\ i = 1, \dots, n$ are $(n \times 1)$ constant vectors. The necessary condition for the extremum of Eq. (5.16) is

$$\nabla Z[X^j, u_j, \lambda_0^j, \lambda_1^j] = 0. \tag{5.17}$$

Finally we have subject function as folloving

$$min(max) Z = min(max) Z_1 + \cdots + min(max) Z_m. \tag{5.18}$$

We now use necessary conditions to find switching points as following

$$x^{j-1}(t_j) = x^j(t_j),\ j = 1, \dots, m, \tag{5.19}$$

conditions in Eq. (5.19), can be expressed as

$$d^j = x^{j-1}(t_j) - x^j(t_j) = 0,\ j = 1, \dots, m, \tag{5.20}$$

$$d^j = (d_i^j),\ i = 1, \dots, n,\ j = 1, \dots, m. \tag{5.21}$$

Now we find switching points by solving Eqs. (5.19) and (5.20) which is nonlinear system.

5.2. **Performance index approximation for the variational problem.** Now we want to extension $x^{(n)}(t)$ in terms of the Bezier polynomials. So let

$$x(t) = X^T \Phi_n(t), \tag{5.22}$$

whereX^T is vector of order $1 \times (n+1)$, By derivating Eq. (5.22) with respect to t we get

$$x^{(1)}(t) = X^T D_\Phi \Phi_n(t), \tag{5.23}$$

where D_Φ is operational matrix of derivative given in Eq. (2.5). By n derivaton of (5.22) with respect to t we have

$$x^{(n)}(t) = X^T D_\Phi^n \Phi_n(t). \tag{5.24}$$

By extending $(t-t_0)^i, i=0,1,\ldots,n-1$ in terms of Bezier polynomials as

$$(t-t_0)^i = d_i\Phi_n(t), \quad i=0,1,\ldots,n-1, \tag{5.25}$$

where $d_i, i=0,1,\ldots,n-1$, are constant vectors of order $1\times(n+1)$ given by

$$d_i = \frac{1}{\binom{n}{i}(t_f-t_0)^{n-i}}[0,\ldots,\binom{i}{i},\binom{i+1}{i},\ldots,\binom{n}{i}], \quad i=0,1,\ldots,n-1, \tag{5.26}$$

So Eq. (4.5) can be rewritten as

$$Z[x(t)] = Z[X]. \tag{5.27}$$

The boundary conditions in Eqs. (4.6) and (4.7) can be expressed as

$$r_k^0 = x^{(k)}(a) - a_k = 0, \quad k=0,\ldots,n-1, \tag{5.28}$$

$$r_k^1 = x^{(k)}(b) - b_k = 0, \quad k=0,\ldots,n-1. \tag{5.29}$$

We now find the extremum of Eq. (4.5) subject to Eqs. (5.28) and (5.29) by using the Lagrange multiplier method. Let

$$Z[x,\lambda] = Z[x,\lambda^0,\lambda^1] + \lambda^0R^0 + \lambda^1R^1, \tag{5.30}$$

where $R^0 = (r_k^0)$, $k=1,\ldots,n$ and $R^1 = (r_k^1)$, $k=1,\ldots,n$ are $(n\times 1)$ constant vectors. The necessary conditions for the extremum of Eq. (5.30) are

$$\nabla Z[X,\lambda^0,\lambda^1] = 0. \tag{5.31}$$

6. Illustrative examples

This section is devoted to numerical examples. We implemented the proposed method in last section with MATLAB (2012). To illustrate our method, we present four numerical examples, and make a comparison with some of the results in the literatures.

Example 1. This example is adapted from [17]

$$\min\ Z = \int_0^5 |u(t)|dt, \tag{6.1}$$

subject to

$$\dot{x}_1(t) = x_2(t) - u(t), \tag{6.2}$$
$$\dot{x}_2(t) = u(t), \tag{6.3}$$
$$|u(t)| \le 1, \tag{6.4}$$

with the boundary conditions as

$$x_1(0) = \frac{1}{2},\ x_2(0) = 1, \tag{6.5}$$
$$x_1(5) = 0, \quad x_2(5) = 0. \tag{6.6}$$

Here we solve this problem with Bezier polynomials by choosing $n = 2$. Let

$$x_1(t) = \Phi_2^T(t)X_1, \tag{6.7}$$

$$x_2(t) = \Phi_2^T(t)X_2, \tag{6.8}$$

$$u(t) = \begin{cases} -1, & [0, t_1], \\ \ \ 0, & [t_1, t_2], \\ \ \ 1, & [t_2, 5], \end{cases} \tag{6.9}$$

$$\tag{6.10}$$

where

$$X_1 = [X_1^0, X_1^1, X_1^2]^T, \qquad X_2 = [X_2^0, X_2^1, X_2^2]^T. \tag{6.11}$$

By using Eqs. (5.16)-(5.18) and Eqs. (6.5)-(6.11) for j=1 and considering interval $[0, t_1]$ we obtain

$$\begin{aligned}
t_1 &= 3 - \sqrt{2},\\
X_1^1 &= [\tfrac{1155}{5809}, \tfrac{248}{299}, \tfrac{1153}{1201}],\\
X_2^1 &= [\tfrac{1121}{2819}, \tfrac{197}{2392}, \tfrac{-577}{2477}],\\
x_1^1(t) &= \tfrac{1155}{5809}B_{0,2}(t) + 2\tfrac{248}{299}B_{1,2}(t) + \tfrac{1153}{1201}B_{2,2}(t) = -\tfrac{1}{2}t^2 + 2t + \tfrac{1}{2},\\
x_2^1(t) &= \tfrac{1121}{2819}B_{0,2}(t) + 2\tfrac{197}{2392}B_{1,2}(t) - \tfrac{577}{2477}B_{2,2}(t) = -t + 1,\\
Z_1 &= t_1 = 3 - \sqrt{2},
\end{aligned}$$

which is exact solution. For j=2 and considering interval $[t_1, t_2]$ we obtain

$$\begin{aligned}
t_2 &= 3 + \sqrt{2},\\
X_1^2 &= [\tfrac{1138}{3771}, \tfrac{357}{1801}, \tfrac{119}{1257}],\\
X_2^2 &= [\tfrac{-102}{1393}, \tfrac{-102}{1393}, \tfrac{-102}{1393}],\\
x_1^2(t) &= \tfrac{1138}{3771}B_{0,2}(t) + 2\tfrac{357}{1801}B_{1,2}(t) + \tfrac{119}{1257}B_{2,2}(t) = \tfrac{1}{36028797018963968}t^2 - \tfrac{577}{985}t + \tfrac{3975}{1189},\\
x_2^2(t) &= -\tfrac{102}{1393}B_{0,2}(t) - 2\tfrac{102}{1393}B_{1,2}(t) - \tfrac{102}{1393}B_{2,2}(t) = -\tfrac{577}{985},\\
Z_2 &= 0.
\end{aligned}$$

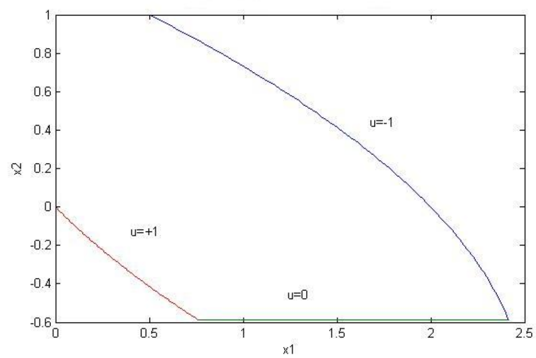
Where the exact solution as following

$$\begin{aligned}
x_1^{2*}(t) &= (\sqrt{2} - 2)t + 9 - 4\sqrt{2} = -\tfrac{577}{985}t + \tfrac{3975}{1189},\\
x_2^{2*}(t) &= 1 - t_1 = \sqrt{2} - 2 = -\tfrac{577}{985}.
\end{aligned}$$

For j=3 and considering interval $[t_2, 5]$ we obtain

$$\begin{aligned}
t_2 &= 3 + \sqrt{2},\\
X_1^3 &= [\tfrac{2174}{985}, \tfrac{1189}{1393}, 0],\\
X_2^3 &= [\tfrac{-985}{577}, \tfrac{-1189}{1393}, 0],\\
x_1^3(t) &= \tfrac{2174}{985}B_{0,2}(t) + 2\tfrac{1189}{1393}B_{1,2}(t) = \tfrac{1}{2}t^2 - 6t + \tfrac{35}{2},\\
x_2^3(t) &= -\tfrac{-985}{577}B_{0,2}(t) - 2\tfrac{1189}{1393}B_{1,2}(t) = t - 5,\\
Z_3 &= 5 - t_2 = 2 - \sqrt{2}.
\end{aligned}$$

FIGURE 1. Plot of state optimal trajectory with bang-bang control for example 1



Finally value of object function is $Z = Z_1 + Z_2 + Z_3 = 5 - 2\sqrt{2}$.

Example 2. This example is adapted from [17]

$$\min\ Z = \int_0^{t_f} (|4 + u(t)|)dt, \tag{6.12}$$

subject to

$$\dot{x}_1(t) = x_2(t), \tag{6.13}$$

$$\dot{x}_2(t) = u(t), \tag{6.14}$$

$$|u(t)| \leq 1. \tag{6.15}$$

with the boundery conditions as

$$x_1(0) = \frac{1}{2}, \quad x_2(0) = 0, \tag{6.16}$$

$$x_1(t_f) = -\frac{1}{2},\ x_2(t_f) = 0. \tag{6.17}$$

Here we solve this problem with Bezier polynomials by choosing $n = 2$. Let

$$u(t) = \begin{cases} -1, & [0, t_1], \\ 0, & [t_1, t_2], \\ 1, & [t_2, t_f]. \end{cases} \tag{6.18}$$

By using Eqs. (6.7) , (6.8) and (6.11) for j=1 and considering interval $[0, t_1]$ we obtain

$$\begin{aligned} t_1 &= \tfrac{\sqrt{6}}{3}, \\ X_1^1 &= [\tfrac{3}{4}, \tfrac{3}{4}, \tfrac{1}{4}], \\ X_2^1 &= [0, \tfrac{-1079}{1762}, \tfrac{-1079}{881}]. \end{aligned}$$

$$x_1^1(t) = \tfrac{3}{4}B_{0,2}(t) + 2\tfrac{3}{4}B_{1,2}(t) + \tfrac{1}{4}B_{2,2}(t) = -\tfrac{1}{2}t^2 + \tfrac{1}{2},$$
$$x_2^1(t) = -2\tfrac{1079}{1762}B_{1,2}(t) - \tfrac{1079}{881}B_{2,2}(t) = -t,$$
$$Z_1 = 5t_1 = 5\tfrac{\sqrt{6}}{3},$$

which is exact silution. For j=2 and considering interval $[t_1, t_2]$ we obtain

$$t_2 = \tfrac{\sqrt{6}}{2},$$
$$X_1^2 = [1, \tfrac{1}{1125899906842624}, -1],$$
$$X_2^2 = [\tfrac{-4801}{980}, \tfrac{-4801}{980}, \tfrac{-4801}{980}],$$
$$x_1^2(t) = B_{0,2}(t) + 2\tfrac{1}{1125899906842624}B_{1,2}(t) - B_{2,2}(t) = -\tfrac{1}{562949953421312}t^2 - \tfrac{\sqrt{6}}{3}t + \tfrac{5}{6},$$
$$x_2^2(t) = -\tfrac{4801}{980}B_{0,2}(t) - 2\tfrac{4801}{980}B_{1,2}(t) - \tfrac{4801}{980}B_{2,2}(t) = -\tfrac{\sqrt{6}}{3},$$
$$Z_2 = 4(\tfrac{\sqrt{6}}{2} - \tfrac{\sqrt{6}}{3}).$$

Where the exact solution as following

$$x_1^{2*}(t) = -\tfrac{\sqrt{6}}{3}t + \tfrac{5}{6},$$
$$x_2^{2*}(t) = -t_1 = -\tfrac{\sqrt{6}}{3}.$$

For j=3 and considering interval $[t_2, t_3]$ we obtain

$$t_f = t_3,$$
$$t_3 = t_1 + t_2 = 5\tfrac{\sqrt{6}}{6},$$
$$X_1^3 = [\tfrac{-1}{4}, \tfrac{-3}{4}, \tfrac{-3}{4}],$$
$$X_2^3 = [\tfrac{-1079}{881}, \tfrac{-1079}{1762}, 0],$$
$$x_1^3(t) = -\tfrac{1}{4}B_{0,2}(t) - 2\tfrac{3}{4}B_{1,2}(t) - \tfrac{3}{4}B_{2,2}(t) = \tfrac{1}{2}t^2 - 5\tfrac{\sqrt{6}}{6}t + \tfrac{19}{12},$$
$$x_2^3(t) = -\tfrac{1079}{881}B_{0,2}(t) - 2\tfrac{1079}{1762}B_{1,2}(t) = t - 5\tfrac{\sqrt{6}}{6},$$
$$Z_3 = 5(t_3 - t_2) = 5\tfrac{\sqrt{6}}{3},$$
$$Z = Z_1 + Z_2 + Z_3 = 4\sqrt{6},$$

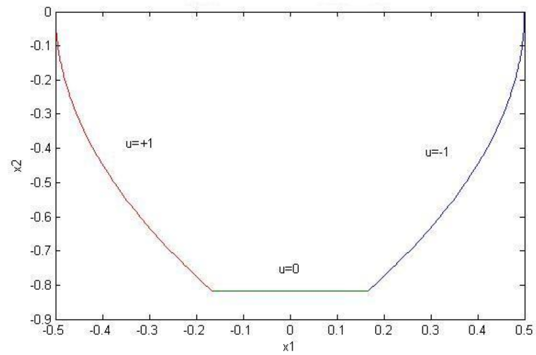
which Z is finally value of object function.

Note that all approximation solutions of example 1 and example 2 are computed with decimal number 16 on intervals $[0, 5]$ and $[0, t_3]$ respectively. Also if you consider another kind of value for control except above value, there is no solution for or if the soluotion exists it would be negative for switching points.

Example 3. To illustrate some of the basic concepts involved when controls are bounded and allowed to have discontinuities we start with a simple physical problem: Derive a controller such that a car moves a distance with a minimum time[22].
The motion equation of the car is as folowing:

$$\min Z = \int_0^{t_f} dt, \tag{6.19}$$

FIGURE 2. Plot of state optimal trajectory with bang-bang control for example 2



subject to

$$\ddot{x}(t) = u, \tag{6.20}$$

$$u = u(t), \ a \leq u \leq b, \tag{6.21}$$

with the boundary conditions of

$$x(0) = 0, \quad \dot{x}(0) = 0, \quad x(t_f) = c, \quad \dot{x}(t_f) = 0. \tag{6.22}$$

We get approximate solutions for the switching and end times as following

$$t_1 = \sqrt{\tfrac{2ac}{b(a+b)}},$$

$$t_f = \sqrt{\tfrac{2c(a+b)}{ab}}.$$

And also we obtain state trejectory and control as folowing

$$u(t) = \begin{cases} b, & [0, t_1], \\ -a, & [t_1, t_f], \end{cases}$$

$$x(t) = \begin{cases} \frac{1}{2}bt^2, & [0, t_1], \\ \\ -\frac{1}{2}a(t - t_f)^2 + c, & [t_1, t_f], \end{cases}$$

which is the exact solution. It is observed that the state trajectory is depended on b and a, c, t_f respectively in intervals $[0, t_1]$ and $[t_1, t_f]$.

7. Conclusion

In this paper we presented a numerical scheme for solving bang-bang optimal control problems with linear constraint. In example 1, the Bezier polynomials were employed in interval $[0, t_f]$ where t_f and end point of $x(t)$ are known. In examples 2 and 3, the Bezier polynomials were employed in interval $[0, t_f]$ where t_f is unknown but the end point of $x(t)$ is known, $u(t)$ is bounded between a and b. Several test problems were used to see the applicability and efficiency of the method. The obtained results show that the new approach can solve the problem effectively. It was investigated to solve bang-bang optimal control problems with a nonlinear constraint.

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