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Multi–soliton solutions, bilinear Bäcklund transformations and Lax pair of nonlinear evolution equation in (2+1)–dimension

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Abstract As an application of Hirota bilinear method, perturbation expansion truncated at different levels is used to obtain exact soliton solutions to (2+1)–dimensional nonlinear evolution equation in much simpler way in comparison to other existing methods. We have derived bilinear form of nonlinear evolution equation and using this bilinear form, bilinear Bäcklund transformations and construction of associated linear problem or Lax pair are presented in straightforward manner and finally for proposed nonlinear equation, explicit one, two and three soliton solutions are also obtained.

Keywords. Soliton solutions, Bilinear Bäcklund transformations, Lax pairs, Perturbation expansion.

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1. Introduction

Advancement in science and mathematical modelling dealing with complex natural phenomenon give rise to induction of more and more nonlinear partial differential equations(NLPDE) with constant and variable coefficients, more precisely with concept of deterministic chaos, natural world has been revealed as nonlinear one [1]. Due to this inherited nonlinearity in partial differential equation, the integrability is big issue, and moreover the integrability for nonlinear partial differential equation cannot be claimed in general. When one says the model is integrable, one should point out under what special meaning it is integrable. Bäcklund transformations, Lax pairs, conservation laws and infinite symmetries may regarded as predictors of complete integrability [23]. For example, absence of movable singularities in solution of equation can be claimed as integrability in sense of Painlevé property and equation having multi-soliton solution is also eligible candidate for complete integrability in term of Hirota's formalism.
Apart from these, integrability can also be defined in terminology of Calogero and

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Eckhaus [4], first is: linearization of equation through local Cole–Hopf transformation and second: when linearization of equation is possible through inverse scattering transform.
Authors of this paper believes on basis of literature survey of several recent publications that existence of multi–soliton solutions for equation can also be regarded as indicator of complete integrability. The inverse scattering technique [5] developed by researchers at Princeton University in 1967 is a powerful tool to analyse the integrability of equation, but it uses powerful analytical methods and makes strong assumptions about equation. On the other hand, exact solution to almost all equation can be obtained by travelling wave reductions, but it would be too simple to claim for integrability.
So in 1971, Ryogo Hirota developed an algorithmic technique [10] (for review see e.g. [8, 9, 15]) to obtain soliton solutions for nonlinear evolution equations. He successfully applied his method to several nonlinear evolution equations including KdV equation, modified KdV equation, Sine–Gordon equation, Kadomtsev–Petviashvili equation, Hirota–Satsuma shallow water wave equation [11–13]. This method popularly known as Hirota bilinear method, serves as powerful method for generating multi–soliton solutions for equation by writing the equation in bilinear form. Once the bilinear form of equation is written one can easily construct soliton solutions of equation, but writing equation in bilinear form is a very tedious job and sometime not possible without inclusion of additional constraints.
In order to apply Hirota method successfully one has to find dependent variable transformation which relies on WTC method [22] and using this transformation equation can be transformed to bilinear form and once the bilinear form is written, exact soliton solution can be obtained by typical perturbation expansion method without the use of inverse scattering technique [15]. Sometime one may face with equation which can not be directly written in bilinear form, in that case new independent variable may be introduced along with subsidiary constraint [17].

In this paper we consider (2+1)–dimensional Jimbo–Miwa equation

$$u_{xxxy} + 3u_y u_{xx} - 3u_{xx} + 3u_x u_{xy} + 2u_{yt} = 0 \tag{1.1}$$

integrability of this equation is already proved by using WTC method [3]. We propose to explore this equation for multi–soliton solutions, bilinear Bäcklund transformations and Lax pair by using Hirota's bilinear method.
This paper is planned as follow: in section 2 detailed discussion about derivation of multi–soliton solution is given, in section 3 we have derived bilinear form, bilinear

Bäcklund transformations and Lax pair for equation (1.1) and in section 4, one, two and three soliton solutions have been derived for equation (1.1). Finally, in section 5, we offered conclusion.

2. Multi Soliton Solution

In this section we would like to give detailed derivation of multi-soliton solutions using perturbation expansion, we shall see how simple ansätz give rise to terminating perturbation expansion when Hirota operator is applied to it.

2.1. **Perturbation expansion.** Let the partial differential equation be written in the form $P[u, u_x, u_t, u_{xx}, u_{xt}, ...] = 0$ such that corresponding bilinear form is written as $F(D)\{f \cdot f\} = 0$ by using dependent variable transformation $u = U[f(x,t)]$. Then for exact soliton solution we take perturbation expansion as suggested in [15]

$$f = 1 + \sum_{i=1}^{N} \epsilon^i f_1 \tag{2.1}$$

such that

$$\begin{aligned} f \cdot f = {} & 1 \cdot 1 + (f_1 \cdot 1 + 1 \cdot f_1)\,\epsilon + (f_2 \cdot 1 + f_1 \cdot f_1 + 1 \cdot f_2)\,\epsilon^2 \\ & + (f_3 \cdot 1 + f_2 \cdot f_1 + f_1 \cdot f_2 + 1 \cdot f_3)\,\epsilon^3 + ... \end{aligned} \tag{2.2}$$

substituting (2.2) into bilinear form $F(D)\{f.f\} = 0$, various coefficients of ϵ equal to zero can be listed as under:

$$\begin{aligned} & \epsilon^0 : F(D)\{1 \cdot 1\} = 0 \text{ which is trivially zero,} \\ & \epsilon^1 : F(D)\{f_1 \cdot 1 + 1 \cdot f_1\} = 2F(D)\{f \cdot 1\} = 0, \\ & \epsilon^2 : 2F(D)\{f_2 \cdot 1\} + F(D)\{f_1 \cdot f_1\} = 0, \\ & \epsilon^3 : 2F(D)\{f_3 \cdot 1\} + 2F(D)\{f_1 \cdot f_2\} = 0, \ \text{...... and so on.} \end{aligned}$$

Generally, using perturbation method, the expansion (2.1) continues to infinite order in ϵ and for exact soliton solution this expansion have to truncate at finite level. Therefore, one may point out that solution so obtained will be just approximation to exact solution. But using this perturbation expansion for bilinear equation and with suitable selection of f_1 makes the perturbation expansion to truncate upto finite sums and therefore solution obtained in such case will be exact one [16]. In next section we demonstrate how to obtain multi–soliton solution using perturbation expansion.

2.2. **One soliton solution.** For one soliton solution we make use of ansätz

$$f_1 = e^{\eta_1}, \quad \eta_1 = \alpha_1 x + \beta_1 y + \gamma_1 t + \delta_1 \tag{2.3}$$

where f_1 assumed to be exponential since base of solitary wave is not superposition of plane waves [16] and parameter α_1 indicates the amplitude of wave and δ_1 which is also called phase constant determines the position of wave.
For one soliton solution it is assumed that $f_i = 0$ for $i \geq 2$, substituting (2.1) into bilinear form and noting down the coefficients of ϵ, coefficient of ϵ^1 gives

$$2F(D)\{f_1 \cdot 1\} = 2F(D)\{e^{\eta_1} \cdot 1\} = 2F(\partial)e^{\eta_1} = 0 \tag{2.4}$$

this gives relation between α_1, β_1 and γ_1 called dispersion relation and in wake of this dispersion relation, the assumption $f_i = 0$ for $i \geq 2$ can be justified [16] and the coefficient of ϵ^2

$$\begin{aligned} &F(D)\{f_1 \cdot f_1\} \\ &= F(D)\{e^{\eta_1} \cdot e^{\eta_1}\} \\ &= F(\alpha_1 - \alpha_1, \beta_1 - \beta_1, \gamma_1 - \gamma_1)e^{\eta_1+\eta_1} \\ &= 0 \end{aligned} \tag{2.5}$$

which is identically zero and other coefficients also vanish because $f_i = 0$ for $i \geq 2$. Since ϵ can be absorbed into phase constant δ_1 of η_1, so without loss of generality by taking $\epsilon = 1$ the perturbation expansion becomes

$$f = 1 + e^{\eta_1}, \quad \eta_1 = \alpha_1 x + \beta_1 + \gamma_1 t + \delta_1$$

and thus one soliton solution is written as follows:

$$u(x,t) = U\left[1 + e^{\eta_1}\right].$$

2.3. **Two soliton solution.** For two soliton solution we make use of ansätz

$$f_1 = e^{\eta_1} + e^{\eta_2}, \quad \eta_i = \alpha_i x + \beta_i y + \gamma_i t + \delta_i, \quad i = 1, 2 \tag{2.6}$$

where in perturbation expansion (2.1) $f_i = 0$ for $i \geq 3$, substituting perturbation expansion (2.2) into $F(D)\{f \cdot f\} = 0$ along with f_1 defined by (2.6) and equating coefficients if ϵ to zero, from coefficient of ϵ^1 we have

$$2F(D)\{f_1 \cdot 1\} = 2F(\partial)\{e^{\eta_1} + e^{\eta_2}\} = 0$$

this gives dispersion relation $F(\alpha_i, \beta_i, \gamma_i) = 0$ for $i = 1, 2$ and the coefficient of ϵ^2 reads as

$$\begin{aligned} F(D)\{f_2 \cdot 1 + f_1 \cdot f_1 + 1 \cdot f_2\} &= 2F(\partial)f_2 + 2F(D)\{e^{\eta_1} \cdot e^{\eta_2}\} \\ &= 2F(\partial)f_2 + 2F(\alpha_1 - \alpha_2, \beta_1 - \beta_2, \gamma_1 - \gamma_2)e^{\eta_1+\eta_2} \\ &= 0 \end{aligned} \tag{2.7}$$

where one must have $f_2 = A_{12}e^{\eta_1+\eta_2}$, and on back substituting f_2 into (2.7).

$$A_{12} = -\frac{F(\alpha_1 - \alpha_2, \beta_1 - \beta_2, \gamma_1 - \gamma_2)}{F(\alpha_1 + \alpha_2, \beta_1 + \beta_2, \gamma_1 + \gamma_2)} \tag{2.8}$$

and coefficients of ϵ^3 is zero and of ϵ^4 vanish identically.
The two soliton solution for perturbation parameter $\epsilon = 1$ is thus written as

$$u = U[1 + e^{\eta_1} + e^{\eta_2} + A_{12}e^{\eta_1+\eta_2}] \tag{2.9}$$

2.4. **Three soliton solution.** As done in previous cases we make use of ansätz

$$f_1 = e^{\eta_1} + e^{\eta_2} + e^{\eta_3}, \quad \eta_i = \alpha_i x + \beta_i y + \gamma_i t + \delta_i, \quad i = 1, 2, 3 \tag{2.10}$$

where in perturbation expansion (2.1) $f_i = 0$ for $i \geq 4$, substituting perturbation expansion (2.2) along with f_1 defined by (2.10) in bilinear form and equating coefficient of ϵ^1 to zero we finds that

$$2F(D)\{f_1 \cdot 1\} = 2F(\partial)\{e^{\eta_1} + e^{\eta_2} + e^{\eta_3}\} = 0$$

this again gives dispersion relation $F(\alpha_i, \beta_i, \gamma_i) = 0$ for $i = 1, 2, 3$. coefficient of ϵ^2 gives the relation

$$F(D)\{f_2 \cdot 1 + 1 \cdot f_2 + f_1 \cdot f_1\} = 2F(\partial)f_2 + F(D)\{f_1 \cdot f_1\} = 0 \tag{2.11}$$

where

$$\begin{aligned} F(D)\{f_1 \cdot f_1\} = F(D)\{e^{\eta_1} \cdot e^{\eta_1} + e^{\eta_2} \cdot e^{\eta_2} + e^{\eta_3} \cdot e^{\eta_3}\} + 2F(D)\left\{\sum_{i \neq j} e^{\eta_i + \eta_j}\right\} \\ = \sum_{i=1}^{3} F(p_i - p_i)e^{2\eta_i} + 2F(p_1 - p_2)e^{\eta_1+\eta_2} \\ + 2F(p_1 - p_3)e^{\eta_1+\eta_3} + 2F(p_2 - p_3)e^{\eta_2+\eta_3} \end{aligned} \tag{2.12}$$

where $F(p_i - p_j) = F(\alpha_i - \alpha_j, \beta_i - \beta_j, \gamma_i - \gamma_j)$ for $i, j = 1, 2, 3$ and thus by using (2.12) in (2.11) one must realise that

$$f_2 = A_{12}e^{\eta_1+\eta_2} + A_{13}e^{\eta_1+\eta_3} + A_{23}e^{\eta_2+\eta_3} \tag{2.13}$$

back substituting f_2 from equation (2.13) into (2.11), it is readily found that

$$A_{ij} = -\frac{F(p_i - p_j)}{F(p_i + p_j)} \tag{2.14}$$

for $i, j = 1, 2, 3$ and $i < j$. Thus f_2 is completely determined and the coefficient of ϵ^3 which gives the relation

$$2F(\partial)f_3 + 2F(D)\{f_1.f_2\} = 0 \tag{2.15}$$

which after some calculation gives

$$-F(\partial)f_3 = [A_{12}F(p_3 - p_1 - p_2) + A_{13}F(p_2 - p_1 - p_3) + A_{23}F(p_1 - p_2 - p_3)]e^{\eta_1+\eta_2+\eta_3} \tag{2.16}$$

from equation (2.16) one can readily guess that

$$f_3 = Be^{\eta_1+\eta_2+\eta_3} \tag{2.17}$$

from (2.17) back substituting f_3 into (2.16) we finds that

$$B = -\frac{A_{12}F(p_3 - p_1 - p_2) + A_{13}F(p_2 - p_1 - p_3) + A_{23}F(p_1 - p_2 - p_3)}{F(p_1 + p_2 + p_3)} \tag{2.18}$$

in order that the coefficient of ϵ^4 should vanish we finds that

$$B = A_{12}A_{13}A_{23} \tag{2.19}$$

combining (2.18) and (2.19) we obtain condition for existence of three soliton solution

$$\begin{aligned} &A_{12}A_{13}A_{23}F(p_1 + p_2 + p_3) + A_{12}F(p_3 - p_1 - p_2) \\ &\quad + A_{13}F(p_2 - p_1 - p_3) + A_{23}F(p_1 - p_2 - p_3) = 0 \end{aligned} \tag{2.20}$$

and coefficients of ϵ^5 and ϵ^6 also vanishes identically. Thus when condition (2.20) is satisfied, the three soliton solution of equation (1.1) can be written as

$$u(x, y, t) = U[1 + e^{\eta_1} + e^{\eta_2} + e^{\eta_3} + A_{12}e^{\eta_1+\eta_2} + A_{13}e^{\eta_1+\eta_3} + A_{23}e^{\eta_2+\eta_3} + Be^{\eta_1+\eta_2+\eta_3}]$$

where B is given by equation (2.19).

3. Bilinearization of equation (1.1)

3.1. **Bilinear form.** As described in [22], by truncating the Laurent expansion about singularity manifold at constant level one can easily obtain Bäcklund transformation for equation (1.1) in the form

$$u = 2(\log f)_x + u_1 \tag{3.1}$$

by taking seed solution $u_1 = 0$, we obtain dependent variable transformation

$$u = 2(\log f)_x \tag{3.2}$$

which transforms the equation (1.1) into quadratic expression in f and its derivatives

$$f_{xxxy}f - f_{xxx}f_y - 3f_{xxy}f_x + 3f_{xx}f_{xy} - 3f_{xx}f + 3f_x^2 + 2f_{yt}f - 2f_tf_y = 0 \tag{3.3}$$

where constant of integration is taken as zero. Using definition of Hirota D-operator, the equation (3.3) readily reduce to bilinear form

$$(D_x^3D_y - 3D_x^2 + 2D_yD_t)f.f = 0 \tag{3.4}$$

which is obtained in corrected form of bilinear equation as given in [3].

3.2. **Bilinear Bäcklund transformations and Lax pair.** The Bäcklund transformations relates the two different solutions of equation, once this transformation is known, starting with seed solution one can construct sequence of explicit solutions for equation. In order to obtain bilinear Bäcklund transformations for equation (1.1) we consider pair of solutions of bilinear form (3.4)

$$(D_x^3D_y - 3D_x^2 + 2D_yD_t)f \cdot f = 0$$
$$(D_x^3D_y - 3D_x^2 + 2D_yD_t)g \cdot g = 0$$

as given in [14] the primitive form of bilinear Bäcklund transformations can be written as

$$[(D_x^3D_y - 3D_x^2 + 2D_yD_t)f \cdot f]g^2 - f^2[(D_x^3D_y - 3D_x^2 + 2D_yD_t)g \cdot g] = 0 \tag{3.5}$$

using exchange formulae for $D_x^3 D_y$, D_x^2 and $D_y D_t$ [14],

$$\begin{aligned}
&(D_x^3 D_y f \cdot f) g^2 - f^2 (D_x^3 D_y g \cdot g) \\
&= 2D_y(D_x^3 f \cdot g) \cdot fg - 6D_x(D_x D_y f \cdot g) \cdot (D_x f \cdot g) \\
&(D_x^2 f \cdot f) g^2 - f^2 (D_x^2 g \cdot g) = 2D_x(D_x f \cdot g) \cdot fg \\
&(D_y D_t f \cdot f) g^2 - f^2 (D_y D_t g \cdot g) = 2D_y(D_t f \cdot g) \cdot fg
\end{aligned}$$

equation (3.5) simplifies to

$$\begin{aligned}
&\{2D_y(D_x^3 f \cdot g) \cdot fg - 6D_x(D_x D_y f \cdot g) \cdot (D_x f \cdot g)\} \\
&\quad + 2\{2D_y(D_t f \cdot g) \cdot fg\} - 3\{2D_x(D_x f \cdot g) \cdot fg)\} = 0
\end{aligned} \tag{3.6}$$

if we take $D_x D_y f.g = \lambda fg$, the equation (3.6) readily decouple into pair

$$(D_x^3 + 2D_t) f \cdot g = 0 \tag{3.7}$$

$$D_x D_y f \cdot g = \lambda fg \tag{3.8}$$

this pair of equations (3.7) and (3.8) is nothing but bilinear form of Bäcklund transformations for equation (1.1). As mentioned by author in [20] the decoupling of (3.5) is not unique, one may also explore for other possible forms of Bäcklund transformations by employing different exchange formulae for Hirota D-operator.

In order to construct Lax pair for equation (1.1), we have to linearise the bilinear Bäcklund transformations (3.7) and (3.8) by suitable transformation, so we introduce new function $\psi(x, y, t)$ through Darboux transformation

$$f = \psi g \tag{3.9}$$

substituting (3.9) into (3.7) and (3.8) and using following formulas [18]

$$\begin{aligned}
&D_x^3 \psi g \cdot g = \psi_{xxx} g^2 + 3\psi_x D_x^2 g \cdot g \\
&D_t \psi g \cdot g = \psi_t g^2 \\
&D_x D_y \psi g \cdot g = \psi_{xy} g^2
\end{aligned}$$

and under dependent variable transformation

$$u = 2(\log g)_x, \quad \frac{D_x^2 g.g}{f^2} = u_x, \quad \frac{D_x D_y g.g}{g^2} = u_y$$

Lax pair for (1.1) are found to be

$$\psi_{xy} + \psi(u_y - \lambda) = 0 \tag{3.10a}$$

$$\psi_{xxx} + 3\psi_x u_x + 2\psi_t = 0 \tag{3.10b}$$

where λ is spectral parameter, one can derive equation (1.1) by eliminating ψ from (3.10) by using compatibility condition $\psi_{xyt} = \psi_{txy}$ provided the condition $\lambda = 1$ is satisfied. This restriction on spectral parameter does not effect the genuinity of Lax pair (3.10). Perhaps this restriction on spectral parameter happened to due decoupling of Eq.(3.6) which according to Ref. [20] can be carried out in infinitely many ways. We have tried other possible decouplings of Eq.(3.6) but unfortunately we could not be successful. In fact, in construction of Lax for nonlinear PDEs, the Lax pair is actually associated eigen value problem where spectral parameter plays role of eigen value and it actually represents integrals of motions for given PDE such that the solution of the original equation plays the role of a potential in the eigenvalue problem. Thus the solution of PDE is mapped to scattering data(values of spectral parameter λ) of the eigenvalue equation. This scattering data in our case is restricted to unity only. But still the procedure of inverse scattering transform can be applied.

4. Multi–soliton solutions

In this section we will construct multi–soliton solutions of equation (1.1) by Hirota method. As this equation is shown to be completely integrable by author in [3], so we may expect existence of multi–soliton solutions for this equation. We will construct its one, two and three soliton solutions.

4.1. **One soliton solution.**

For one soliton solution we take

$$f = 1 + \epsilon f_1, \quad f_1 = e^{\eta_1}, \quad \eta_1 = \alpha_1 x + \beta_1 y + \gamma_1 t + \delta \tag{4.1}$$

where α_1 and β_1 are wave numbers and γ_1 being frequency, substituting (4.1) into bilinear form (3.4)

$$(D_x^3 D_y - 3D_x^2 + 2D_y D_t) f \cdot f = 0$$

from coefficient of ϵ, the dispersion law follows as

$$\alpha_1^3 \beta_1 - 3\alpha_1^2 + 2\beta_1\gamma_1 = 0 \tag{4.2}$$

and coefficient of ϵ^2 is identically zero, thus by taking perturbation parameter $\epsilon = 1$ and making use dependent variable transformation (3.2), the one soliton solution to equation (1.1) is found to be

$$u(x, y, t) = \frac{2\alpha_1 e^{\eta_1}}{1 + e^{\eta_1}}, \quad \eta_1 = \alpha_1 x + \beta_1 y + \left(\frac{3\alpha_1^2}{2\beta_1} - \frac{\alpha_1^3}{2}\right) t + \delta_1 \tag{4.3}$$

and this satisfies equation (1.1).

4.2. **Two soliton solution.** For two soliton solution we take

$$f = 1 + \epsilon f_1 + \epsilon^2 f_2, \quad f_1 = e^{\eta_1} + e^{\eta_2}, \quad \eta_i = \alpha_i x + \beta_i y + \gamma_i t + \delta_i, \tag{4.4}$$

here $i = 1, 2$, substituting (4.4) into bilinear form (3.4), the dispersion law from coefficient of ϵ follows as

$$\alpha_i^3 \beta_i - 3\alpha_i^2 + 2\beta_i \gamma_i = 0, \quad i = 1, 2 \tag{4.5}$$

from coefficient of ϵ^2, we have

$$f_2 = A_{12} e^{\eta_1 + \eta_2} \tag{4.6}$$

where

$$A_{12} = \frac{\alpha_1^3\beta_2 + 3\alpha_1^2\alpha_2\beta_1 - 3\alpha_1^2\alpha_2\beta_2 - 3\alpha_1\alpha_2^2\beta_1 + 3\alpha_1\alpha_2^2\beta_2 + \alpha_2^3\beta_1 - 6\alpha_1\alpha_2 + 2\beta_1\gamma_2 + 2\beta_2\gamma_1}{\alpha_1^3\beta_2 + 3\alpha_1^2\alpha_2\beta_1 + 3\alpha_1^2\alpha_2\beta_2 + 3\alpha_1\alpha_2^2\beta_1 + 3\alpha_1\alpha_2^2\beta_2 + \alpha_2^3\beta_1 - 6\alpha_1\alpha_2 + 2\beta_1\gamma_2 + 2\beta_2\gamma_1} \tag{4.7}$$

coefficient of ϵ^3 is identically zero. Thus for perturbation parameter $\epsilon = 1$, the two soliton solution for (1.1) can be written as

$$f = 1 + e^{\eta_1} + e^{\eta_2} + A_{12} e^{\eta_1 + \eta_2} \tag{4.8}$$

where A_{12} can be interpreted as phase shift function, that is, during interaction of solitons, a phase shift A_{12} is introduced [2] and solution (4.8) can be regarded as nonlinear superposition of two solitons [19] and finally by using dependent variable transformation (3.2) we write

$$u(x,t) = \frac{2[\alpha_1 e^{\eta_1} + \alpha_2 e^{\eta_2} + A_{12}(\alpha_1 + \alpha_2)e^{\eta_1 + \eta_2}]}{1 + e^{\eta_1} + e^{\eta_2} + A_{12} e^{\eta_1 + \eta_2}} \tag{4.9}$$

where $\eta_i = \alpha_i x + \beta_i y + \left(\frac{3\alpha_i^2}{2\beta_i} - \frac{\alpha_i^3}{2}\right) t + \delta_i$ for $i = 1, 2$ and A_{12} can be calculated as above, this solution also solves equation (1.1) and can be verified by direct substitution. If an equation can be put in a bilinear form, then certainly it must have atleast two soliton solutions [6], however, existence of three soliton solution is subjected to verification of three soliton solution condition as we shall see in next section.

4.3. **Three soliton solution.** An equation which can be written in bilinear form will definitely possess one and two soliton solution, but when one talk about three soliton solution, it occurs in very restrictive manner [7] and it can be verified that for equation (1.1), the restriction (2.20) for three soliton solution is satisfied. On similar line of action, for three soliton solution we assume

$$\begin{aligned} &f = 1 + \epsilon f_1 + \epsilon^2 f_2 + \epsilon^3 f_3, \quad f_1 = e^{\eta_1} + e^{\eta_2} + e^{\eta_3}, \\ &\eta_i = \alpha_i x + \beta_i y + \gamma_i t + \delta_i \end{aligned} \tag{4.10}$$

for $i = 1, 2, 3$, substituting (4.10) into bilinear form (3.4) and from coefficients of ϵ we realise that the dispersion law is

$$\alpha_i^3 \beta_i - 3\alpha_i^2 + 2\beta_i \gamma_i = 0, \quad i = 1, 2, 3 \tag{4.11}$$

and

$$f_2 = A_{12} e^{\eta_1+\eta_2} + A_{13} e^{\eta_1+\eta_3} + A_{23} e^{\eta_2+\eta_3} \tag{4.12}$$

$$f_3 = B e^{\eta_1+\eta_2+\eta_3} \tag{4.13}$$

where form of A_{12} is already given at (4.7) and

$$A_{13} = \frac{\begin{matrix}\alpha_1^3\beta_3 + 3\alpha_1^2\alpha_3\beta_1 - 3\alpha_1^2\alpha_3\beta_3 - 3\alpha_1\alpha_3^2\beta_1 + 3\alpha_1\alpha_3^2\beta_3 \\ 1 \qquad + \alpha_3^3\beta_1 - 6\alpha_1\alpha_3 + 2\beta_1\gamma_3 + 2\beta_3\gamma_1\end{matrix}}{\begin{matrix}\alpha_1^3\beta_3 + 3\alpha_1^2\alpha_3\beta_1 + 3\alpha_1^2\alpha_3\beta_3 + 3\alpha_1\alpha_3^2\beta_1 + 3\alpha_1\alpha_3^2\beta_3 \\ 1 \qquad + \alpha_3^3\beta_1 - 6\alpha_1\alpha_3 + 2\beta_1\gamma_3 + 2\beta_3\gamma_1\end{matrix}}$$

$$A_{23} = \frac{\begin{matrix}\alpha_2^3\beta_3 + 3\alpha_2^2\alpha_3\beta_2 - 3\alpha_2^2\alpha_3\beta_3 - 3\alpha_2\alpha_3^2\beta_2 + 3\alpha_2\alpha_3^2\beta_3 \\ 1 \qquad + \alpha_3^3\beta_2 - 6\alpha_2\alpha_3 + 2\beta_2\gamma_3 + 2\beta_3\gamma_2\end{matrix}}{\begin{matrix}\alpha_2^3\beta_3 + 3\alpha_2^2\alpha_3\beta_2 + 3\alpha_2^2\alpha_3\beta_3 + 3\alpha_2\alpha_3^2\beta_2 + 3\alpha_2\alpha_3^2\beta_3 \\ 1 \qquad + \alpha_3^3\beta_2 - 6\alpha_2\alpha_3 + 2\beta_2\gamma_3 + 2\beta_3\gamma_2\end{matrix}}$$

and B is given by (2.19) and finally by taking perturbation parameter $\epsilon = 1$ we have

$$f = 1 + (e^{\eta_1} + e^{\eta_2} + e^{\eta_3}) + (A_{12} e^{\eta_1+\eta_2} + A_{13} e^{\eta_1+\eta_3} + A_{23} e^{\eta_2+\eta_3}) + B e^{\eta_1+\eta_2+\eta_3} \tag{4.14}$$

and thus by using dependent variable transformation (3.2), the three soliton solution for equation (1.1) is found to be

$$\begin{aligned} u = 2[&\alpha_1 e^{\eta_1} + \alpha_2 e^{\eta_2} + \alpha_3 e^{\eta_3} \\ &+ (A_{12}(\alpha_1+\alpha_2) e^{\eta_1+\eta_2} + A_{13}(\alpha_1+\alpha_3) e^{\eta_1+\eta_3} + A_{23}(\alpha_2+\alpha_3) e^{\eta_2+\eta_3}) \\ &+ B(\alpha_1+\alpha_2+\alpha_3) e^{\eta_1+\eta_2+\eta_3}][1 + (e^{\eta_1} + e^{\eta_2} + e^{\eta_3}) \\ &+ (A_{12} e^{\eta_1+\eta_2} + A_{13} e^{\eta_1+\eta_3} + A_{23} e^{\eta_2+\eta_3}) + B e^{\eta_1+\eta_2+\eta_3}]^{-1} \end{aligned} \tag{4.15}$$

for $\eta_i = \alpha_i x + \beta_i y + \left(\frac{3\alpha_i^2}{2\beta_i} - \frac{\alpha_i^3}{2}\right) t + \delta_i$ for $i = 1, 2, 3$, where by direct substitution of (4.15) into equation (1.1), one can verify the existence of three soliton solution for equation. But in principle, the existence of three soliton solution is week condition to claim for complete integrability [7], however, existence of three such solution is strong indicator for existence of N-soliton solutions and existence of N-soliton solutions is

sign of complete integrability in sense of Hirota's formalism [9](see also [21]). Since equation (1.1) also passes Painlevé property [3] so existence of N-soliton solutions is again justified. However, one may also go with conclusion of Ramani [21], that Painlevé property and three soliton solution go hand in hand and existence of three soliton solution is suffice to claim for complete integrability.
For N-soliton solution one can extend the expansion (4.10)

$$f = \sum_{\mu_i \in \{0,1\}} \exp\left\{ \sum_{0<i,j\leq N} A'_{ij}\mu_i\mu_j + \sum_{i=1}^{N} \eta_i\mu_i \right\}, \quad A_{ij} = \exp\left(A'_{ij}\right) \tag{4.16}$$

and if existence of $(N-1)$-soliton solution is assumed, then one can obtain condition [7]

$$\sum_{\sigma_i=\pm 1} F\left[\sum_{i=1}^{N} \sigma_i p_i\right] \prod_{0<i<j\leq N} [F(\sigma_i p_i - \sigma_j p_j)\sigma_i\sigma_j] = 0 \tag{4.17}$$

the equation (4.17) is called Hirota's condition for existence of N-soliton solutions and it automatically hold for $N = 1, 2$ and for $N = 3$ this equation reduces to (2.20) which has been verified by direct computation.

5. Conclusion

To conclude, we have firmly established the complete integrability of (2+1)–dimensional nonlinear evolution equation, the given equation is known to be completely integrable in sense of Painlevé property and existence of three soliton solution, where three soliton condition is also satisfied, that, again confirms the complete integrability in sense of Hirota's formalism. In addition to this, we have also derived Lax pair and Bäcklund transformations directly using Hirota method.

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