



## Existence and uniqueness of positive and nondecreasing solution for nonlocal fractional boundary value problem

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**Abstract** In this article, we verify the existence and uniqueness of a positive and nondecreasing solution for nonlinear boundary value problem of fractional differential equation in the form

$$D_{0+}^{\alpha} x(t) + f(t, x(t)) = 0, \quad 0 < t < 1, \quad 2 < \alpha \leq 3,$$
$$x(0) = x'(0) = 0, \quad x'(1) = \beta x(\xi),$$

where  $D_{0+}^{\alpha}$  denotes the standard Riemann-Liouville fractional derivative,  $0 < \xi < 1$  and  $0 < \beta \xi^{\alpha-1} < \alpha - 1$ . Our analysis relies on the fixed point theorem in partially ordered sets. An illustrative example is also presented.

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**Keywords.** Boundary value problem, Fixed point theorem, Partially ordered set, Positive solution, non-decreasing solution.

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### 1. INTRODUCTION

In recent years, fractional calculus is one of the interesting issues that have attracted the attention of many scientists, specially in mathematics and engineering sciences. Many natural phenomena can be presented by boundary value problems of fractional differential equations. Many authors in different fields such as chemical physics, fluid flows, electrical networks, viscoelasticity, try to model the phenomena by boundary value problems of fractional differential equations [1-4]. In order to achieve extra information in fractional calculus, specially boundary value problems, the reader can refer to more valuable papers or books that are written by authors

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[5-26].

Bai [17] discussed the existence of positive solutions for the BVP

$$\begin{aligned} D_{0+}^{\alpha} u(t) + f(t, u(t)) &= 0, \quad 0 < t < 1, \quad 1 < \alpha \leq 2, \\ u(0) &= 0, \quad D^{\alpha-1} u(1) = \beta u(\eta), \quad 0 < \eta < 1, \end{aligned}$$

where  $\alpha$  is a real number,  $0 < \beta \eta^{\alpha-1} < 1$  and  $D_{0+}^{\alpha}$  denotes Riemann-Liouville fractional derivative.

In [14], Zhang used some fixed point theorems on cones to verify the existence of a positive solution for the equation

$$D_{0+}^{\alpha} u(t) + f(t, u(t)) = 0, \quad 0 < t < 1, \quad 1 < \alpha \leq 2,$$

with the boundary conditions  $u(0) + u'(0) = u(1) + u'(1) = 0$ .

El-Shahed [27] studied the existence and nonexistence of positive solutions to the following nonlinear FDE

$$D_{0+}^{\alpha} u(t) + \lambda f(u(t)) = 0, \quad 0 < t < 1, \quad 2 < \alpha \leq 3,$$

along with boundary conditions  $u(0) = u'(0) = u'(1) = 0$ , where  $D_{0+}^{\alpha}$  denotes standard Riemann-Liouville fractional derivative.

In this paper, which is motivated by paper [23], we investigate the existence and uniqueness of a positive and nondecreasing solution for a nonlocal boundary value problem for fractional differential equation of the form

$$D_{0+}^{\alpha} x(t) + f(t, x(t)) = 0, \quad 0 < t < 1, \quad 2 < \alpha \leq 3, \quad (1.1)$$

$$x(0) = x'(0) = 0, \quad x'(1) = \beta x(\xi), \quad (1.2)$$

where  $D_{0+}^{\alpha}$  denotes standard Riemann-Liouville fractional derivative,  $0 < \xi < 1$ ,  $0 < \beta \xi^{\alpha-1} < \alpha - 1$ , and  $f \in C([0, 1] \times [0, \infty), [0, \infty))$ . The main approach is based upon a fixed point theorem in partially ordered sets.

## 2. BACKGROUND MATERIALS AND PRELIMINARIES

We now give definitions, lemmas and theorems that will be used in the remainder of this paper.

**Definition 2.1.** ([7,8]) The Riemann-Liouville fractional integral of order  $\alpha > 0$ , of a function  $x : \mathbb{R}^{>0} \rightarrow \mathbb{R}$  is defined by

$$I_{0+}^{\alpha} x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x(s) ds, \quad n-1 < \alpha \leq n, \quad (2.1)$$



provided that right-hand side is point wise defined on  $\mathbb{R}^{>0}$ .

**Definition 2.2.** ([7,8]) The Riemann-Liouville fractional derivative of order  $\alpha > 0$ , of a function  $x : \mathbb{R}^{>0} \rightarrow \mathbb{R}$  is given by

$$D_{0+}^{\alpha}x(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha-1} x(s) ds, \quad n-1 < \alpha \leq n. \quad (2.2)$$

**Lemma 2.3.** ([7,8]) Let  $x \in C(0,1) \cap L(0,1)$ . Then fractional differential equation  $D_{0+}^{\alpha}x(t) = 0$  has

$$x(t) = a_1 t^{\alpha-1} + a_2 t^{\alpha-2} + \dots + a_n t^{\alpha-n}, \quad a_i \in \mathbb{R}, \quad i = 1, \dots, n, \quad n = [\alpha] + 1, \quad (2.3)$$

as a unique solution.

**Lemma 2.4.** ([7,8]) Let  $x \in C(0,1) \cap L(0,1)$  with a fractional derivative of order  $\alpha > 0$  that belongs to  $C(0,1) \cap L(0,1)$ . Then

$$I_{0+}^{\alpha} D_{0+}^{\alpha} x(t) = x(t) + a_1 t^{\alpha-1} + a_2 t^{\alpha-2} + \dots + a_n t^{\alpha-n}, \quad (2.4)$$

for some  $a_i \in \mathbb{R}$ ,  $i = 1, \dots, n$ ,  $n = [\alpha] + 1$ .

**Theorem 2.5.** ([28]) Assume that  $(K, \leq)$  be a partially ordered set. Suppose that there exists a metric  $d$  in  $K$  so that  $(K, d)$  is a complete metric space satisfying the following condition:

$$\text{if } (u_n) \text{ is a nondecreasing sequence in } K \text{ s.t } u_n \rightarrow u, \text{ then } u_n \leq u, \quad \forall n \in \mathbb{N}. \quad (2.5)$$

Let  $T : K \rightarrow K$  be a nondecreasing mapping so that

$$d(Tu, Tv) \leq d(u, v) - \varphi(d(u, v)),$$

where  $\varphi : \mathbb{R}^{\geq 0} \rightarrow \mathbb{R}^{\geq 0}$  is a continuous and nondecreasing function such that  $\varphi$  is positive in  $\mathbb{R}^{>0}$ ,  $\varphi(0) = 0$  and  $\lim_{t \rightarrow \infty} \varphi(t) = \infty$ . If there exists  $u_0 \in K$  with  $u_0 \leq Tu_0$ , then  $T$  has a fixed point.

**Theorem 2.6.** ([28]) Assume that hypotheses of Theorem 2.5 are satisfied. Suppose that  $(K, \leq)$  satisfies the following condition:

$$\text{for } x, y \in K \text{ there exists } z \in K \text{ which is comparable to } x \text{ and } y. \quad (2.6)$$

Then, the fixed point is unique.

**NOTE.** In this paper, we use real Banach space  $K = C[0,1]$  with the norm  $\|u\| = \max_{t \in [0,1]} \{u(t)\}$ . This space can be equipped with a partial order given by

$$u, v \in C[0,1], \quad u \leq v \Leftrightarrow u(t) \leq v(t), \quad t \in [0,1].$$



Nieto and Rodriguez [29] proved  $(K, \leq)$  with the metric  $d(u, v) = \sup_{t \in [0, 1]} \{|u(t) - v(t)|\}$  satisfied condition (2.5). Also, for  $u, v \in C[0, 1]$  if  $\max\{u, v\} \in K$ , then,  $(K, \leq)$  satisfies condition (2.6).

**Lemma 2.7.** *Assume that  $0 < \xi < 1$ ,  $\frac{\alpha-1}{\beta} \neq \xi^{\alpha-1}$  and  $g(t) \in C[0, 1]$ . Then, fractional boundary value problem*

$$D_{0+}^{\alpha} x(t) + g(t) = 0, \quad 0 < t < 1, \quad 2 < \alpha \leq 3, \quad (2.7)$$

$$x(0) = x'(0) = 0, \quad x'(1) = \beta x(\xi), \quad (2.8)$$

has a unique solution

$$x(t) = \int_0^1 G_1(t, s)g(s)ds + \frac{\beta t^{\alpha-1}}{\alpha-1-\beta\xi^{\alpha-1}} \int_0^1 G_2(\xi, s)g(s)ds, \quad (2.9)$$

where

$$G_1(t, s) = \frac{1}{\Gamma(\alpha)} \begin{cases} t^{\alpha-1}(1-s)^{\alpha-2} - (t-s)^{\alpha-1}, & 0 \leq s \leq t \leq 1, \\ t^{\alpha-1}(1-s)^{\alpha-2}, & 0 \leq t \leq s \leq 1, \end{cases} \quad (2.10)$$

$$G_2(\xi, s) = \frac{1}{\Gamma(\alpha)} \begin{cases} \xi^{\alpha-1}(1-s)^{\alpha-2} - (\xi-s)^{\alpha-1}, & 0 \leq s \leq \eta \leq 1, \\ \xi^{\alpha-1}(1-s)^{\alpha-2}, & 0 \leq \eta \leq s \leq 1, \end{cases} \quad (2.11)$$

*Proof.* Lemma 2.4 guarantees that

$$x(t) = a_1 t^{\alpha-1} + a_2 t^{\alpha-2} + a_3 t^{\alpha-3} - I_{0+}^{\alpha} g(t). \quad (2.12)$$

From boundary conditions (2.8), we obtain  $a_2 = a_3 = 0$  and

$$a_1 = \frac{\alpha-1}{\Gamma(\alpha)(\alpha-1-\beta\xi^{\alpha-1})} \int_0^1 (1-s)^{\alpha-2} g(s)ds - \frac{\beta}{\Gamma(\alpha)(\alpha-1-\beta\xi^{\alpha-1})} \int_0^{\xi} (\xi-s)^{\alpha-1} g(s)ds.$$

Substituting  $a_1, a_2$  and  $a_3$  into (2.12), we get

$$\begin{aligned} x(t) &= -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s)ds + \frac{(\alpha-1)t^{\alpha-1}}{\Gamma(\alpha)(\alpha-1-\beta\xi^{\alpha-1})} \int_0^1 (1-s)^{\alpha-2} g(s)ds \\ &\quad - \frac{\beta t^{\alpha-1}}{\Gamma(\alpha)(\alpha-1-\beta\xi^{\alpha-1})} \int_0^{\xi} (\xi-s)^{\alpha-1} g(s)ds. \end{aligned}$$



Thus, the unique solution of FBVP (2.7)-(2.8) is

$$\begin{aligned}
 x(t) &= \frac{-1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s) ds + \frac{t^{\alpha-1}}{\Gamma(\alpha)} \left( \int_0^t (1-s)^{\alpha-2} g(s) ds + \int_t^1 (1-s)^{\alpha-2} g(s) ds \right) \\
 &+ \frac{\beta t^{\alpha-1}}{\Gamma(\alpha)(\alpha-1-\beta\xi^{\alpha-1})} \left( \xi^{\alpha-1} \int_0^1 (1-s)^{\alpha-2} g(s) ds - \int_0^\xi (\xi-s)^{\alpha-1} g(s) ds \right) \\
 &= \frac{1}{\Gamma(\alpha)} \int_0^t [t^{\alpha-1}(1-s)^{\alpha-2} - (t-s)^{\alpha-1}] g(s) ds + \frac{1}{\Gamma(\alpha)} \int_t^1 t^{\alpha-1}(1-s)^{\alpha-2} g(s) ds \\
 &+ \frac{\beta t^{\alpha-1}}{\Gamma(\alpha)(\alpha-1-\beta\xi^{\alpha-1})} \int_0^\xi [\xi^{\alpha-1}(1-s)^{\alpha-2} - (\xi-s)^{\alpha-1}] g(s) ds \\
 &+ \frac{\beta t^{\alpha-1}}{\Gamma(\alpha)(\alpha-1-\beta\xi^{\alpha-1})} \int_\xi^1 \xi^{\alpha-1}(1-s)^{\alpha-2} g(s) ds \\
 &= \int_0^1 G_1(t,s) g(s) ds + \frac{\beta t^{\alpha-1}}{\alpha-1-\beta\xi^{\alpha-2}} \int_0^1 G_2(\xi,s) g(s) ds.
 \end{aligned}$$

The proof is complete. □

**Lemma 2.8.** *The function  $G_1(t, s)$  is nonnegative and continuous function.*

*Proof.* It is easy to see that  $G_1$  is continuous. Now, we show that  $G_1(t, s) \geq 0$ .

Case 1.  $s = 1$ . Obviously  $G_1(t, 1) = 0$

Case 2. For  $0 \leq t \leq s \leq 1$ , it is clear that  $G_1(t, 1) = t^{\alpha-1}(1-s)^{\alpha-2} \geq 0$ .

Case 3. For  $0 \leq s \leq t \leq 1$ , we have

$$\begin{aligned}
 G_1(t, s) &= \frac{1}{\Gamma(\alpha)} [t^{\alpha-1}(1-s)^{\alpha-2} - (t-s)^{\alpha-1}] \\
 &= \frac{1}{\Gamma(\alpha)} [t^{\alpha-1} \frac{(1-s)^{\alpha-1}}{(1-s)} - (t-s)^{\alpha-1}] \\
 &\geq \frac{1}{\Gamma(\alpha)} [t^{\alpha-1}(1-s)^{\alpha-1} - (t-s)^{\alpha-1}] \\
 &= \frac{1}{\Gamma(\alpha)} [(t-ts)^{\alpha-1} - (t-s)^{\alpha-1}] \geq 0.
 \end{aligned}$$

Thus, the lemma is proved. □

**Lemma 2.9.** *For  $G_1$  and  $G_2$ , the following relations are satisfied, respectively,*

$$\text{(i)} \sup_{t \in [0,1]} \int_0^1 G_1(t, s) ds = \frac{1}{(\alpha-1)\Gamma(\alpha+1)}, \quad \text{(ii)} \int_0^1 G_2(\xi, s) ds = \frac{\xi^{\alpha-1}}{\Gamma(\alpha)} \left( \frac{1}{\alpha-1} - \frac{\xi}{\alpha} \right).$$



*Proof.* (i) We have

$$\begin{aligned}\int_0^1 G_1(t, s) ds &= \frac{1}{\Gamma(\alpha)} \left\{ \int_0^t t^{\alpha-1} (1-s)^{\alpha-2} ds - \int_0^t (t-s)^{\alpha-1} ds + \int_t^1 t^{\alpha-1} (1-s)^{\alpha-2} ds \right\} \\ &= \frac{1}{\Gamma(\alpha)} \left\{ \frac{t^{\alpha-1}}{\alpha-1} - \frac{t^\alpha}{\alpha} \right\}.\end{aligned}$$

Let us set

$$k(t) = \frac{1}{\Gamma(\alpha)} \left\{ \frac{t^{\alpha-1}}{\alpha-1} - \frac{t^\alpha}{\alpha} \right\}.$$

Then, as

$$k'(t) = \frac{1}{\Gamma(\alpha)} \{ t^{\alpha-2} - t^{\alpha-1} \} > 0,$$

$k(t)$  is strictly increasing and this follows that

$$\sup_{t \in [0,1]} \int_0^1 G_1(t, s) ds = k(1) = \frac{1}{(\alpha-1)\Gamma(\alpha+1)}.$$

(ii) We have

$$\begin{aligned}\int_0^1 G_2(\xi, s) ds &= \frac{1}{\Gamma(\alpha)} \left\{ \int_0^\xi \xi^{\alpha-1} (1-s)^{\alpha-2} ds - \int_0^\xi (\xi-s)^{\alpha-1} ds + \int_\xi^1 \xi^{\alpha-1} (1-s)^{\alpha-2} ds \right\} \\ &= \frac{\xi^{\alpha-1}}{\Gamma(\alpha)} \left( \frac{1}{\alpha-1} - \frac{\xi}{\alpha} \right).\end{aligned}$$

Therefore, the proof is completed.  $\square$

**Lemma 2.10.** *The function  $G_1(t, s)$  is strictly increasing with respect to the first component.*

*Proof.* We assume that

$$g_1(t) = \frac{1}{\Gamma(\alpha)} [t^{\alpha-1} (1-s)^{\alpha-2} - (t-s)^{\alpha-1}], \quad s \leq t,$$

$$g_2(t) = \frac{1}{\Gamma(\alpha)} t^{\alpha-1} (1-s)^{\alpha-2}, \quad t \leq s,$$

where  $s$  is fixed. For  $s \leq t$ , we have

$$\begin{aligned}g_1'(t) &= \frac{1}{\Gamma(\alpha)} [(\alpha-1)t^{\alpha-2} (1-s)^{\alpha-2} - (\alpha-1)(t-s)^{\alpha-2}] \\ &= \frac{1}{\Gamma(\alpha-1)} [(t-ts)^{\alpha-2} - (t-s)^{\alpha-2}] \geq 0.\end{aligned}$$

Then,  $g_1(t)$  is strictly increasing on  $[s, 1]$ . On other hand, for  $0 \leq t_1 < t_2 \leq s$ , it is easy to see that  $g_2(t_1) < g_2(t_2)$  and so,  $g_2(t)$  is strictly increasing on  $[0, s]$ . Therefore,



when  $t_1 < t_2 \leq s$ , and  $s \leq t_1 < t_2$ , we have  $G_1(t_1, s) < G_1(t_2, s)$ . Now, we show that in the case  $t_1 \leq s \leq t_2$  with  $t_1 < t_2$ ,  $G_1(t_1, s) < G_1(t_2, s)$ . We have  $g_2(t_1) \leq g_2(s) = g_1(s) \leq g_1(t_2)$ . If  $g_2(t_1) = g_1(t_2)$ , then  $g_2(t_1) = g_2(s) = g_1(s) = g_1(t_2)$  and since  $g_1$  and  $g_2$  are monotone, we conclude that  $t_1 = t_2 = s$ , which contradicts with  $t_1 < t_2$ . Then  $g_2(t_1) < g_1(t_2)$  and this shows that  $G_1(t_1, s) < G_1(t_2, s)$ .  $\square$

### 3. MAIN RESULTS

For convenience of presentation, we now present the following hypothesis to be used in the rest of the paper.

(H<sub>1</sub>)  $f : [0, 1] \times \mathbb{R}^{\geq 0} \rightarrow \mathbb{R}^{\geq 0}$  is a continuous function and nondecreasing with respect to second variable.

(H<sub>2</sub>) There exist  $H \subset [0, 1]$  with  $\mu(H) > 0$  so that  $f(t, x(t)) \neq 0$  for  $t \in H$  and  $x \in \mathbb{R}$ .

(H<sub>3</sub>) There exist  $0 < \gamma < \left( \frac{1}{(\alpha-1)\Gamma(\alpha+1)} + \frac{\beta\xi^{\alpha-1}}{(\alpha-1-\beta\xi^{\alpha-1})\Gamma(\alpha)} \left( \frac{1}{\alpha-1} - \frac{\xi}{\alpha} \right) \right)^{-1}$  such that for  $x, y \in \mathbb{R}^{\geq 0}$ , with  $x \leq y$

$$f(t, x) - f(t, y) \leq \gamma\kappa(x - y), \quad 0 \leq t \leq 1,$$

where  $\kappa(x) = \frac{x}{x+2}$ .

**Theorem 3.1.** *Assume that the hypotheses (H<sub>1</sub>)-(H<sub>3</sub>) hold. Then, the boundary value problem (1.1)-(1.2) has a unique positive and strictly increasing solution  $x(t)$ .*

*Proof.* Consider the cone

$$K = \{x \in C[0, 1] : x(t) \geq 0\}.$$

Since  $K$  is a closed set,  $K$  is a complete metric space with the distance given by

$$d(x, y) = \sup_{t \in [0, 1]} |x(t) - y(t)|.$$

Let us define the operator  $\mathfrak{A} : C[0, 1] \rightarrow C[0, 1]$ , by

$$\mathfrak{A}x(t) = \int_0^1 G_1(t, s)f(s, x(s))ds + \frac{\beta t^{\alpha-1}}{\alpha - 1 - \beta\xi^{\alpha-1}} \int_0^1 G_2(\xi, s)f(s, x(s))ds. \quad (3.1)$$

By Lemma 2.8 and (H<sub>1</sub>), it follows that  $\mathfrak{A}(K) \subseteq K$ .

Now, using Theorems 2.5 and 2.6 and in three steps, we will prove to theorem.

**Step 1. Existence of nonnegative solution.**



By  $(\mathbf{H}_1)$ , for  $x, y \in K$  with  $x \geq y$ , we have

$$\begin{aligned}\mathfrak{A}x(t) &= \int_0^1 G_1(t, s)f(s, x(s))ds + \frac{\beta t^{\alpha-1}}{1 - \beta\xi^{\alpha-2}} \int_0^1 G_2(\xi, s)f(s, x(s))ds \\ &\geq \int_0^1 G_1(t, s)f(s, y(s))ds + \frac{\beta t^{\alpha-1}}{1 - \beta\xi^{\alpha-2}} \int_0^1 G_2(\xi, s)f(s, y(s))ds \\ &= \mathfrak{A}y(t).\end{aligned}$$

This shows that  $\mathfrak{A}$  is nondecreasing operator. Also, for  $x \geq y$ , we have

$$\begin{aligned}d(\mathfrak{A}x, \mathfrak{A}y) &= \sup_{t \in [0,1]} |(\mathfrak{A}x)(t) - (\mathfrak{A}y)(t)| = \sup_{t \in [0,1]} [(\mathfrak{A}x)(t) - (\mathfrak{A}y)(t)] \\ &\leq \sup_{t \in [0,1]} \int_0^1 G_1(t, s)[f(s, x(s)) - f(s, y(s))]ds \\ &\quad + \frac{\beta t^{\alpha-1}}{\alpha - 1 - \beta\xi^{\alpha-1}} \int_0^1 G_2(\xi, s)[f(s, x(s)) - f(s, y(s))]ds \\ &\leq \sup_{t \in [0,1]} \int_0^1 G_1(t, s) \frac{\gamma[x(s) - y(s)]}{x(s) - y(s) + 2} ds \\ &\quad + \frac{\beta}{\alpha - 1 - \beta\xi^{\alpha-1}} \int_0^1 G_2(\xi, s) \left[ \frac{\gamma[x(s) - y(s)]}{x(s) - y(s) + 2} \right] ds.\end{aligned}$$

Using Lemma 2.9 and  $(\mathbf{H}_3)$  and since we know the function  $\kappa(x) = \frac{x}{x+2}$  is nondecreasing, we get

$$\begin{aligned}d(\mathfrak{A}x, \mathfrak{A}y) &\leq \gamma \cdot \frac{\|x - y\|}{\|x - y\| + 1} \left( \sup_{t \in [0,1]} \int_0^1 G_1(t, s)ds + \frac{\beta}{\alpha - 1 - \beta\xi^{\alpha-1}} \int_0^1 G_2(\xi, s)ds \right) \\ &= \gamma \cdot \frac{\|x - y\|}{\|x - y\| + 1} \left( \frac{1}{(\alpha - 1)\Gamma(\alpha + 1)} + \frac{\beta\xi^{\alpha-1}}{(\alpha - 1 - \beta\xi^{\alpha-1})\Gamma(\alpha)} \left( \frac{1}{\alpha - 1} - \frac{\xi}{\alpha} \right) \right) \\ &< \left[ \|x - y\| - \left( \|x - y\| - \frac{\|x - y\|}{\|x - y\| + 1} \right) \right].\end{aligned}$$

Let us set  $\varphi(u) = u - \frac{u}{u+1}$ . It is easy to see that  $\varphi : \mathbb{R}^{\geq 0} \rightarrow \mathbb{R}^{\geq 0}$  is continuous, nondecreasing, positive in  $\mathbb{R}^{> 0}$ ,  $\varphi(0) = 0$  and  $\lim_{u \rightarrow \infty} \varphi(u) = \infty$ . So, for  $x \geq y$ , we have

$$d(\mathfrak{A}x, \mathfrak{A}y) \leq d(x, y) - \varphi(d(x, y)).$$

On the other hand,  $G_1(t, s) \geq 0$  and  $f \geq 0$  yields

$$(\mathfrak{A}0)(t) = \int_0^1 G_1(t, s)f(s, 0)ds \geq 0.$$





Now, all the conditions of Theorem 2.5 are satisfied. Consequently, the BVP (1.1)-(1.2) has at least one nonnegative the solution.

**Step 2. Uniqueness of the solution.**

It is clear that  $(K, \leq)$  satisfies condition 2.6 and so by Theorem 2.6, we conclude the uniqueness of solution.

**Step 3. The solution is strictly increasing.**

As  $x(0) = \int_0^1 G_1(0, s)f(s, x(s))ds$  and  $G_1(0, s) = 0$ , we have  $x(0) = 0$ . We assume that  $t_1, t_2 \in [0, 1]$  with  $t_1 < t_2$ . We distinguish the following cases.

**Case 1.** If  $t_1 = 0$ , then  $x(t_1) = 0$ . We know that  $x(t) \geq 0$ . Assuming  $x(t_2) = 0$ , we get  $G_1(t_2, s)f(s, x(s)) = 0, a.e.(s)$  and since  $G_1(t_2, s) \neq 0$  we have  $f(s, x(s)) = 0, a.e.(s)$ . On the other hand,  $f$  is nondecreasing with respect to the second variable, and so we have  $f(s, 0) \leq f(s, x(s)) = 0, a.e.(s)$ , which contradicts the hypothesis  $(\mathbf{H}_2)$ . Consequently,  $x(t_1) = 0 < x(t_2)$ .

**Case 2.** If  $t_1 > 0$ , then

$$\begin{aligned} x(t_2) - x(t_1) &= (\mathfrak{A}x)(t_2) - (\mathfrak{A}x)(t_1) \\ &+ \int_0^1 [G_1(t_2, s) - G_1(t_1, s)]f(s, x(s))ds \\ &+ \frac{\beta(t_2^{\alpha-1} - t_1^{\alpha-1})}{\alpha - 1 - \beta\xi^{\alpha-1}} \int_0^1 G_2(\eta, s)f(s, x(s))ds. \end{aligned}$$

Now, by Lemma 2.10 and since  $f$  is nonnegative, we get  $x(t_1) = x(t_2)$  and or  $x(t_1) < x(t_2)$ .

Assume that  $x(t_1) = x(t_2)$ . Then

$$\int_0^1 [G_1(t_2, s) - G_1(t_1, s)]f(s, x(s))ds = 0,$$

and this follows that

$$[G_1(t_2, s) - G_1(t_1, s)]f(s, x(s))ds = 0, \quad a.e.(s),$$

and Lemma 2.10 yields

$$f(s, x(s))ds = 0, \quad a.e.(s),$$

which again contradicts the hypothesis  $(\mathbf{H}_2)$ . Thus,  $x(t_1) < x(t_2)$ . Therefore, the proof is completed. □



## 4. AN EXAMPLE

**Example 4.1.** Consider the boundary value problem

$$\begin{cases} D_{0+}^{\frac{5}{2}}x(t) + \frac{tx+x}{4x+8} = 0, & 0 < t < 1, \\ x(0) = x'(0) = 0, & x'(1) = x(\frac{1}{2}), \end{cases} \quad (4.1)$$

in which  $\beta = 1$ ,  $\xi = \frac{1}{2}$ ,  $f(t, x) = \frac{t+1}{4} \cdot \frac{x}{x+2}$  with

$$f(t, x) - f(t, y) = \frac{t+1}{4} \left( \frac{x}{x+2} - \frac{y}{y+2} \right) < \frac{1}{2} \cdot \frac{x-y}{x-y+2}, \quad x \geq y.$$

Furthermore,

$$\left( \frac{1}{(\alpha-1)\Gamma(\alpha+1)} + \frac{\beta\xi^{\alpha-1}}{(\alpha-1-\beta\xi^{\alpha-1})\Gamma(\alpha)} \left( \frac{1}{\alpha-1} - \frac{\xi}{\alpha} \right) \right)^{-1} \simeq 3.2373 > \frac{1}{2} = \gamma.$$

Thus, the conclusion of Theorem 3.1 applies to problem (4.1).

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