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Optimization with the time-dependent Navier-Stokes equations as constraints

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Abstract In this paper, optimal distributed control of the time-dependent Navier-Stokes equations is considered. The control problem involves the minimization of a measure of the distance between the velocity field and a given target velocity field. A mixed numerical method involving a quasi-Newton algorithm, a novel calculation of the gradients and an inhomogeneous Navier-Stokes solver, to find the optimal control of the Navier-Stokes equations is proposed. Numerical examples are given to demonstrate the efficiency of the method.

Keywords. Optimal control problems, Navier-Stokes equations,PDE-constrained optimization, quasi-Newton algorithm, finite difference.

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1. Introduction

Optimal control of the Navier-Stokes equations is a constrained optimization problem with the Navier-Stokes equations serving as subsidiary conditions. PDE-constrained optimization problems arise in many applications and so received much progress during recent years for example Hicken and Alonso [14] described an algorithm for PDE-constrained optimization that controls numerical errors. Akbarian and Keyanpour [1] proposed a new numerical method for solving a class of fractional order optimal control problems. The fractional optimal control theory is a new topic in Mathematics. Kar [17] discussed an optimal control problem in population dynamics.

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In general, optimal control problems are classified into distributed and boundary control problems. In the case of additional constraints, optimal control problems rather than the Navier-Stokes equations are divided into four groups: (i) State-constrained [8], (ii) Control-constrained [25], (iii) mixed state-control constrained [9] and (iv) optimal control of the Navier-Stokes equations without inequality constraints [16], [13]. Hinze and Kunisch [16] have used second order methods for optimal control of the time-dependent Navier-Stokes equations for the first time. Hinze [15] has delivered a good literature review up to 2002 in his research report.

Li et al. [19] have presented a general method based on adjoint formulation for the optimal control of distributed parameter systems and applied it to the problem of controlling vortex shedding behind a cylinder governed by the unsteady two dimensional Navier-Stokes equations with time-dependent boundary conditions. They used a quasi-Newton Daviden-Fletcher-Powell (DFP) method for the minimization of the cost function. Chaabane et al. [4] have considered a non-convex cost function of the velocity gradient tensor and provide the optimality systems based on a lagrangian formulation and adjoint equations. Optimal boundary control problems for the three dimensional evolutionary Navier-Stokes equations in the exterior of a bounded domain are investigated by Fursikov et al. [10], theoretically. In their work the drag functional is minimized as a control objective. The problem of an appropriate choice of a cost functional for vortex reduction for unsteady flows is considered by Kunisch and Vexler [18]. Brizitskii [3] has studied optimal control problems for the stationary Navier-Stokes equations with mixed boundary conditions on velocity and derived some theorems on the uniqueness and stability of solutions for the particular functional that depend on the total pressure. Bilić in [2] has derived some theoretical results for the optimal control of a coefficient in modification of the Navier-Stokes equations. She considered the coefficient of the kinematic viscosity to be a positive function of the velocity gradient. Recently, the existence of optimal solution and the maximum principle for optimal control problem of Navier-Stokes equations with state constraint of point wise type in three dimensions are studied by Liu [20].

In this work, optimal distributed control of the time-dependent incompressible Navier-Stokes equations in the absence of inequality constraints is considered. The tracking type objective functional is minimized in order to control the behaviour of the fluid flow close enough to the specific target flow. While the continuous optimality conditions are given, the discretization of the optimality system is derived. The discretized system is solved by a quasi-Newton algorithm [21] using a novel calculation of the gradients [11] and an inhomogeneous Navier-Stokes solver.

The paper is organized as follows. In Section 2, we will briefly review some theory in optimal control of the Navier-Stokes equations. Section 3 gives details of the optimization method applied for the optimal control problem and deals with numerical solution of the inhomogeneous Navier-Stokes equations. Numerical results are given in Section 4.

2. Optimal control of Navier-Stokes equations

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with $\partial\Omega \in \mathcal{C}^2$. Solenoidal spaces are introduced as follows

$$H := \overline{\{v \in C_0^{\infty\,2} : \mathrm{div} v = 0\}}^{|\cdot|_{L^2(\Omega)^2}},$$
$$V := \overline{\{v \in C_0^{\infty\,2} : \mathrm{div} v = 0\}}^{|\cdot|_{H^1(\Omega)^2}},$$

where the superscripts denote closures in the respective norms. Optimal distributed control of the Navier-Stokes equations is given by [15]

$$\begin{aligned}
&\min_{(\boldsymbol{u},\boldsymbol{f})\in W\times U} \quad J(\boldsymbol{u},\boldsymbol{f}),\\
&\qquad s.t.:\\
&\qquad\quad \frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \frac{1}{Re}\Delta\boldsymbol{u} + \nabla p = \boldsymbol{f}, \quad in\,\Omega\times(0,T),\\
&\qquad\quad \nabla\cdot\boldsymbol{u} = 0, \quad in\,\Omega\times(0,T),\\
&\qquad\quad \boldsymbol{u}(t,\cdot) = 0, \quad on\,(\partial\Omega)\times(0,T),\\
&\qquad\quad \boldsymbol{u}(0,\cdot) = \boldsymbol{u}_0, \quad in\,\Omega,
\end{aligned} \tag{2.1}$$

where $\boldsymbol{u} : [0,T]\times\Omega \to \mathbb{R}^2$ is the velocity field, $p : [0,T]\times\Omega \to \mathbb{R}$ is the pressure field, $\boldsymbol{f} : [0,T]\times\Omega \to \mathbb{R}^2$ is the control function, T is the final time, Re is the Reynolds number, $\boldsymbol{u}_0$ is a given initial velocity field and U is the Hilbert space of controls. Further we define W to be

$$W = \{v \in L^2(V) : v_t \in L^2(V^*)\},$$

with the associated norm

$$|v|_W = |v|_{L^2(V)} + |v_t|_{L^2(V^*)},$$

where V^* is the dual space of V.
In this paper, we consider the tracking type objective functional

$$J(\boldsymbol{u},\boldsymbol{f}) = \frac{1}{2}\int_0^T\int_\Omega |\boldsymbol{u}-\boldsymbol{z}|^2 dxdydt + \frac{\alpha}{2}|\boldsymbol{f}|_U^2.$$

Here $\boldsymbol{z}$ is a given velocity of a target flow and α is a penalty parameter. It is assumed that J is bounded from below, weakly lower semi-continuous, twice Fréchet differentiable with locally Lipschitzean second derivative and radially unbounded in $\boldsymbol{f}$.
Thanks to Theorem 2.1 in Ref. [15] for every $\boldsymbol{f} \in U$, there exists a unique element $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{f})$ satisfying the Navier-Stokes equations, therefore problem (2.1) can be equivalently written as

$$\min_{\boldsymbol{f} \in U} \quad \hat{J}(\boldsymbol{f}) = J(\boldsymbol{u}(\boldsymbol{f}), \boldsymbol{f}).$$

Theorem 2.1. *Under above assumptions, problem* (2.1) *admits a solution* $(\boldsymbol{u}^*(\boldsymbol{f}^*), \boldsymbol{f}^*) \in W \times U$.

Proof. See [15]. $\square$

2.1. **Lid driven cavity control.** Let us consider the lid driven cavity control of the Navier-Stokes equations in two dimensions as follows

$$\begin{aligned}
(p): \quad \min J(u, v, f_1, f_2) &= \frac{1}{2}\int_0^1 \int_0^1 \int_0^1 [(u - z_1)^2 + (v - z_2)^2] dxdydt \\
&\quad + \frac{\alpha}{2}\int_0^1 \int_0^1 \int_0^1 (f_1^2 + f_2^2) dxdydt, \\
& s.t.: \\
& u_t + p_x + u \cdot u_x + v \cdot u_y - \frac{1}{Re}(u_{xx} + u_{yy}) = f_1, \quad in\, \Omega \times (0, T), \\
& v_t + p_y + u \cdot v_x + v \cdot v_y - \frac{1}{Re}(v_{xx} + v_{yy}) = f_2, \quad in\, \Omega \times (0, T), \\
& u_x + v_y = 0, \quad in\, \Omega, \\
& u = 1, v = 0, \quad \Gamma_1 : 0 \leq x \leq 1, y = 1, \\
& u = 0, v = 0, \quad \Gamma_2 : 0 \leq x \leq 1, y = 0, \\
& u = 0, v = 0, \quad \Gamma_3 : x = 0, 0 \leq y \leq 1, \\
& u = 0, v = 0, \quad \Gamma_4 : x = 1, 0 \leq y \leq 1, \\
& u(0, \cdot) = 0, \quad in\, \Omega, \\
& v(0, \cdot) = 0, \quad in\, \Omega,
\end{aligned} \tag{2.2}$$

where u and v are the components of the velocity field, z_1 and z_2 are the components of the velocity of the target flow, f_1 and f_2 are the components of the control variable, p is the pressure, Γ_1, Γ_2, Γ_3 and Γ_4 are the corresponding boundaries for $\Omega = (0, 1) \times (0, 1)$.

3. Numerical solution for the optimal control problem

In this section we present a typical optimization algorithm as follows [11].

Algorithm.

Start with an initial guess $\boldsymbol{f}^0 = (f_1^0, f_2^0)$ for the control vector. Then for $n = 0, 1, 2, ...$

(1) Solve the Navier-Stokes equations to obtain the corresponding state $\boldsymbol{u}^n = (u^n, v^n)$,
(2) Compute $\frac{d\hat{J}}{df}|_{\boldsymbol{f}^n}$,
(3) Use the results of steps (1) and (2) to compute an optimal direction $\delta\boldsymbol{f}$,
(4) set $\boldsymbol{f}^{n+1} = \boldsymbol{f}^n + \delta\boldsymbol{f}$.

For the step (1) and step (3) we use a Navier-Stokes solver and the quasi-Newton Broyden-Fletcher-Goldfarb-Shanno (BFGS) algorithm [21], respectively. For the step (2) we use the following method. By the chain rule we have

$$\hat{J}' = \begin{bmatrix} \frac{\partial J}{\partial f_1} \\ \\ \frac{\partial J}{\partial f_2} \end{bmatrix} = \begin{bmatrix} \frac{\partial J_1}{\partial u} \cdot \frac{\partial u}{\partial f_1} + \frac{\partial J_2}{\partial f_1} \\ \\ \frac{\partial J_1}{\partial v} \cdot \frac{\partial v}{\partial f_2} + \frac{\partial J_2}{\partial f_2} \end{bmatrix},$$

where

$$J_1 = \frac{1}{2}\int_0^1\int_0^1\int_0^1 [(u - z_1)^2 + (v - z_2)^2] dxdydt,$$
$$J_2 = \frac{\alpha}{2}\int_0^1\int_0^1\int_0^1 (f_1^2 + f_2^2) dxdydt,$$

The terms $\frac{\partial J_2}{\partial f_1}$, $\frac{\partial J_1}{\partial v}$, $\frac{\partial J_1}{\partial u}$ and $\frac{\partial J_2}{\partial f_2}$ are as follows [15]

$$\frac{\partial J_1}{\partial u} = u - z_1, \quad \frac{\partial J_1}{\partial v} = v - z_2,$$
$$\frac{\partial J_2}{\partial f_1} = \alpha f_1, \quad \frac{\partial J_2}{\partial f_2} = \alpha f_2.$$

For the terms $\frac{\partial u}{\partial f_1}$ and $\frac{\partial v}{\partial f_2}$, we use the finite difference approximation [11]

$$\frac{\partial u}{\partial f_1}|_{f_1^n} = \frac{u(f_1^n) - u(\overline{f_1})}{f_1^n - \overline{f_1}},$$
$$\frac{\partial v}{\partial f_2}|_{f_2^n} = \frac{v(f_2^n) - v(\overline{f_2})}{f_2^n - \overline{f_2}},$$

where $\overline{f_1}$ and $\overline{f_2}$ are some values close to f_1^n and f_2^n, respectively.

3.1. Numerical solution of the Navier-Stokes equations.

For the numerical solution of the Navier-Stokes equations, a finite difference solver given by Seibold [23], mitnavierstokes1806.m, have been extended to inhomogeneous Navier-Stokes equations using Strang's general approach [24]. Now, we present a brief summary of the numerical solution approach. While (u, v) and p are the solutions to the Navier-Stokes equations, we denote the numerical approximations by capital letters. Assume we have the velocity field U^n and V^n at the n^{th} time step and condition (2.2) is satisfied. We find the solution at the $(n+1)^{th}$ time step by the following three step approach:

- Explicit treatment of the nonlinear terms
  $$\frac{U^* - U^n}{\Delta t} = -((U^n)^2)_x - (U^n V^n)_y + {f_1}^n,$$
  $$\frac{V^* - V^n}{\Delta t} = -(U^n V^n)_x - ((V^n)^2)_y + {f_2}^n,$$
- Implicit treatment of the viscosity terms
  $$\frac{U^{**} - U^*}{\Delta t} = \frac{1}{Re}(U^{**}_{xx} + U^{**}_{yy}),$$
  $$\frac{V^{**} - V^*}{\Delta t} = \frac{1}{Re}(V^{**}_{xx} + V^{**}_{yy}),$$
- Pressure correction
  We correct the intermediate velocity field (U^{**}, V^{**}) by the gradient of a pressure P^{n+1} to enforce incompressibility.
  $$\frac{U^{n+1} - U^{**}}{\Delta t} = -(P^{n+1})_x,$$
  $$\frac{V^{n+1} - V^{**}}{\Delta t} = -(P^{n+1})_y,$$
  in vector notation the correction equations read as
  $$\frac{1}{\Delta t}\boldsymbol{U}^{n+1} - \frac{1}{\Delta t}\boldsymbol{U}^{n} = -\nabla P^{n+1}.$$
  Applying the divergence to both sides of the above equation yields the following linear system of equations.
  $$-\Delta P^{n+1} = -\frac{1}{\Delta t}\nabla \cdot \boldsymbol{U}^{n}.$$

4. Numerical results

To elucidate this presentation and test the efficiency of the proposed method, we consider the lid driven cavity optimal control of the Navier-Stokes equations. Let the domain be a unit square, $\Omega = [0, 1] \times [0, 1]$, and the final time, T, to be 4. Spacial discretization is done on a staggered grid. The temporal and spacial mesh sizes are set to $\Delta t = 0.01$ and $\Delta x = \Delta y = 1/76$, respectively.

Example 1. In this example we consider the Stokes flow as the target flow. To do this, the Navier-Stokes equations with a Reynolds number that was very smaller than one is solved as the target flow. A Navier-Stokes flow with Reynolds number equal to 10 matches the target flow in the beginning time steps.
The target flow and the controlled flow at $t = 0$, $t = 0.3$ and $t = 1$ are shown in Figure 1. The figure shows that a good match is achieved at the time t=0.3.
Furthermore, the infinity norm of $(\boldsymbol{U} - \boldsymbol{z})$ between the controlled flow and the target flow and the infinity norm of $\boldsymbol{f}$ of the optimal control versus time is shown in Figure 2.
As one can see the error $\|\boldsymbol{U} - \boldsymbol{z}\|$ goes to zero and the norm of the control becomes relatively large at the beginning in order to steer the controlled flow to the target flow and then after a good match is achieved, its norm remains relatively constant. Table 1 demonstrates the infinity norm of approximate gradient, the difference between the controlled flow and the target flow and the control variable, respectively.

Example 2. Here, we have solved the Navier-Stokes equations with a Reynolds number equal to 100 and obtained the velocity of the target flow. The target flow is a Navier-Stokes flow with Reynolds number equal to 100. In this case, a flow with Reynolds number equal 20 is made close enough to the target flow. Similar results are obtained that is shown in figures 3 and 4 and Table 2.
These examples confirm that the proposed method works well in making close enough a Navier-Stokes flow with low Reynolds number to a Navier-Stokes flow with higher Reynolds number and vice versa.

5. Conclusion

In this article, we have combined several mathematical methods including a quasi-Newton algorithm, a new calculation of the gradients and a Navier-Stokes solver, to solve the optimal control of the time-dependent Navier-Stokes equations. We have extended a homogeneous Navier-Stokes solver to an inhomogeneous one and used it for the optimization problem.

The implementation of this method is simple and gives good results in initial iterations. The numerical results have demonstrated the accuracy of the proposed method.

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Table 1. Performance of the BFGS algorithm for Example 1.

Iteration	$\hat{J}'$	$norm(\boldsymbol{U}-\boldsymbol{z})$	$norm(\boldsymbol{f})$
0	0.061473583412436	0.203901843008942	0.000000000000000
1	0.054389208298978	0.193638540827147	0.061473583412436
2	0.024879628982099	0.146965631884720	0.541876350806589
⋮	⋮	⋮	⋮
11	0.009998960016509	0.116695406569955	0.916886759034144
12	0.008275454355323	0.112942185162138	1.002200080230743
13	0.007538604486665	0.111613872586078	1.021250608012849
⋮	⋮	⋮	⋮
28	0.001328585499258	0.100935354138841	1.140885901732639
29	0.000947337525458	0.100475833390172	1.150904114553860
30	0.000698513544273	0.100104699457118	1.161715541713162
31	0.000634101503420	0.099971732039035	1.166425537183772
32	0.000462344144686	0.099863683095352	1.169826797463214
⋮	⋮	⋮	⋮
44	0.000124632802094	0.099166438586193	1.174530798071361
45	0.000095432962331	0.099107117332099	1.174662484579273
⋮	⋮	⋮	⋮
98	0.000000016334996	0.098989716424033	1.172247512947354
99	0.000000012047153	0.098989718420992	1.172247553027968

FIGURE 1. Streamlines of target flow (top left), streamlines of controlled flow at t=0 (top right), streamlines of controlled flow at t=0.3 (bottom left) and streamlines of controlled flow at t=1 (bottom right) for Example 1.

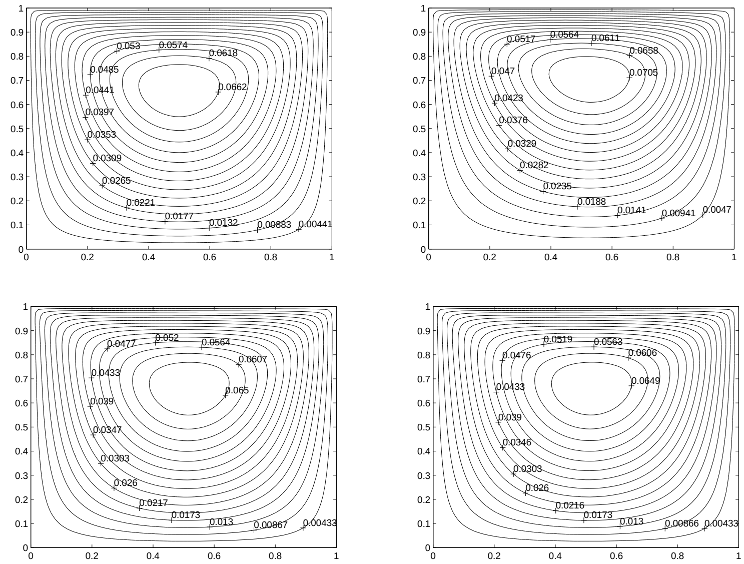

FIGURE 2. The norm $\|\boldsymbol{f}\|$ of the optimal control (left) and the norm $\|\boldsymbol{U}-\boldsymbol{z}\|$ between the controlled flow and the target flow (right) vs. time for Example 1.

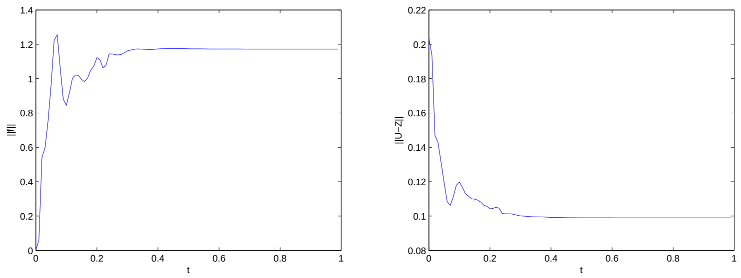

FIGURE 3. Streamlines of target flow (top left), streamlines of controlled flow at t=0 (top right), streamlines of controlled flow at t=1 (bottom left) and streamlines of controlled flow at t=4 (bottom right) for Example 2.

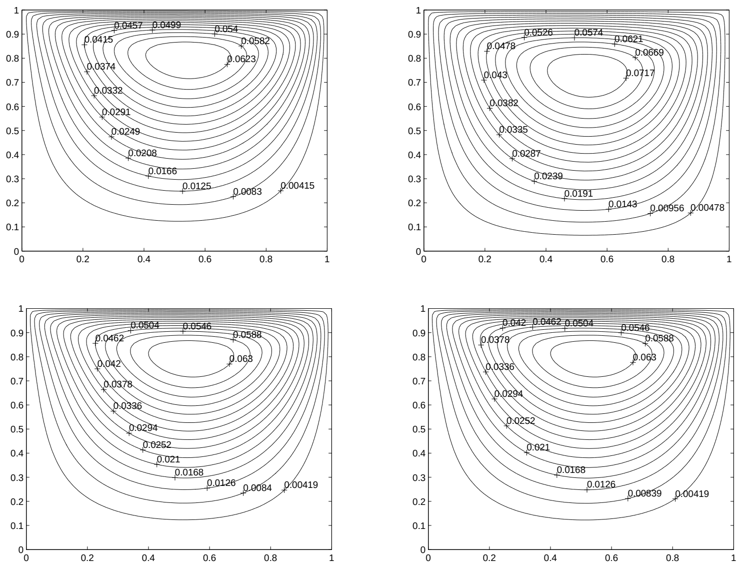

FIGURE 4. The norm $\|\boldsymbol{f}\|$ of the optimal control (left) and the norm $\|\boldsymbol{U} - \boldsymbol{z}\|$ between the controlled flow and the target flow (right) vs. iteration for Example 2.

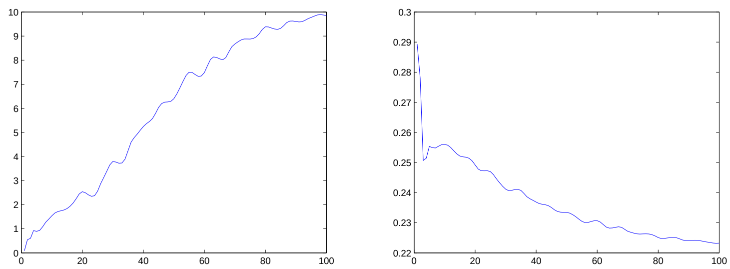

TABLE 2. Performance of the BFGS algorithm for Example 2.

Iteration	$\hat{J}'$	$norm(\boldsymbol{U}-\boldsymbol{z})$	$norm(\boldsymbol{f})$
0	0.081829692380660	0.289374201890752	0.000000000000000
1	0.071082950052867	0.278008875833359	0.081829692380660
2	0.052545279623648	0.250687694252410	0.547936121577062
$\vdots$	$\vdots$	$\vdots$	$\vdots$
200	0.000011492695819	0.221699668533595	10.212535627658578
201	0.000010945474911	0.221699043846042	10.212132358401371
202	0.000009858529245	0.221697181401740	10.211796591651238
$\vdots$	$\vdots$	$\vdots$	$\vdots$
398	0.000000000407023	0.221683839087889	10.206532516428085
399	0.000000000376881	0.221683839073550	10.206532523734090
400	0.000000000327853	0.221683839045965	10.206532536020260