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Brenstien polynomials and applications to fractional differential equations

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Abstract The paper is devoted to the study of Brenstien Polynomials and development of some new operational matrices of fractional order integrations and derivatives. The operational matrices are used to convert fractional order differential equations to systems of algebraic equations. A simple scheme yielding accurate approximate solutions of the couple systems for fractional differential equations is developed. The scheme is designed such a way that it can be easily simulated with any computational software. The efficiency of proposed method verified by some test problems.

Keywords. Brenstien polynomials, Coupled systems, Fractional differential equations, operational matrices of integrations, Numerical simulations.

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1. Introduction

During the last three decades a tremendous amount of work is devoted to the study of fractional calculus and is extensively used in many fields of science and technology. Due to the nonlocal nature of fractional order operators, fractional calculus is extensively used to model many important models of viscoelasticity, see for example [7, 30, 56]. It is also successfully used in modeling important problems of fluid dynamics and solid mechanics [31, 48], nanotechnology [32], bioengineering [8], finance [55] and many more important problems of physics and engineering, see for example [22, 38, 40, 47, 53] and the references quoted there.

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Fractional order differential equation (FDEs), being generalization of integer order differential equations provide more real insight of natural phenomena as compare to the classical integer order counter part. However, due to computational complexities involved in these equations, it is very difficult and some time impossible to obtain exact analytic solutions of FDEs. Hence, it is better to find approximate solutions to the FDEs. The accuracy of approximate solutions solely depends on the selection of available techniques. Many methods are available for solutions of FDEs such as Homotopy perturbation method [23, 37, 58], Variational iteration method [34, 35], Laplace transform [2,29,33,41,54], Differential transform method [3,36], Runge Kutta methods [4, 5], Neural network methods [42, 43] etc.

Spectral method, using orthogonal polynomials as basis functions, is one of the powerful method for approximate solutions of FDEs. Recently, various basis are used and interesting results are produced by many researchers for a variety of problems. For example, in [9, 18, 27] Jacobi polynomials are used and operational matrices are developed to solve FDES and FPDEs. In [11,14,15,24,45,46,52] Legendre polynomials are used and operational matrices are developed along with spectral tau and collocation methods for solutions of some important problems. In [10, 19, 20, 50] shifted Chebyshev polynomials are used to develop efficient techniques for solutions of FDEs and FPDEs. Operational matrices using Laguerre polynomials are also addressed by many authors, interested reader may find the work in [12,13,15,25,39,44,49,51] very useful. Besides, some operational matrices are also developed for solutions of FPDEs in high dimensional space in [16, 26–28].

One of the well known class of polynomials frequently used in literature is the class of Bernstein polynomials. These polynomials are not orthogonal but they have certain properties which makes them applicable to approximation theory, we refer to [1, 6, 17, 21, 57]. These polynomials are enriched with many useful properties, like the form a partition of unity etc. In this paper our approach is based on Bernstein polynomials. We construct new operational matrices of fractional order integration and derivatives. These matrices are then used to provide a theoretical treatment to solve a generalized class of multi term FDEs of the form

$$D^{\alpha}U(t) = \sum_{i=0}^{n} d_i D^{\beta_i} U(t) + g(t), \tag{1.1}$$

where all the derivatives are defined in the Caputo sense, $n \leq \alpha < n+1$, $0 \leq \beta_0 < 1 \leq \beta_1 \cdots < n \leq \beta_n < n+1$, d_i are all real constants, $g(t)$ is source term and $U(t) \in C([0,1])$ is the unknown solution to be determined. The method is designed under the assumption that $U(t)$ and $g(t)$ along with (1.1) satisfies all the necessary conditions for the existence of a unique solution. We then extend the idea to solve coupled system of FDEs of the form

$$\begin{aligned} D^{\alpha_1}U(t) &= \sum_{i=0}^{n} a_i D^{\beta_{1i}}U(t) + \sum_{i=0}^{n} b_i D^{\beta_{2i}}V(t) + g_1(t), \\ D^{\alpha_2}V(t) &= \sum_{i=0}^{n} c_i D^{\gamma_{1i}}U(t) + \sum_{i=0}^{n} d_i D^{\gamma_{2i}}V(t) + g_2(t), \end{aligned} \tag{1.2}$$

where a_i, b_i, c_i, d_i are real constants, $U(t)$, $V(t)$ are unknown solutions to be determined and $g_1(t)$, $g_2(t)$ are the known source terms of the system. The order of derivatives is analogously defined as in (1.1).

The rest of the article is organized as: in section 2, we provide some basic definitions from fractional calculus and Brenstein Polynomials, in section 3, we derive new operational matrices. In section 4, the new matrices are applied to develop a theoretical treatment to FDEs and couple system of FDEs and finally, in section 5 some test problems are solved and a short conclusion is provided.

2. Preliminaries

This section summarizes some basic concepts, definitions and known results from fractional calculus, which are useful for our study. We refer the readers to [47,55] for detail study.

Definition 2.1. Given an interval $[a,b] \subset \mathbf{R}$, the Riemann-Liouville fractional order integral of order $\alpha \in \mathbf{R}_+$ of a function $\phi \in (L^1[a,b], \mathbf{R})$ is defined by

$$\mathcal{I}^{\alpha}_{a+}\phi(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1}\phi(s)ds,$$

provided the integral on right hand side exists.

Definition 2.2. For a given function $\phi(x) \in C^n[a,b]$, the Caputo fractional order derivative of order α is defined as

$$D^{\alpha}\phi(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{\phi^{(n)}(t)}{(x-t)^{\alpha+1-n}} dt, \; n-1 \le \alpha < n \;, n \in N,$$

provided the right side is pointwise defined on (a,∞), where $n = [\alpha]+1$ in case α not an integer and $n = \alpha$ in case α is an integer.

Hence, it follows that

$$D^{\alpha}x^{k} = \frac{\Gamma(1+k)}{\Gamma(1+k-\alpha)}x^{k-\alpha},\ I^{\alpha}x^{k} = \frac{\Gamma(1+k)}{\Gamma(1+k+\alpha)}x^{k+\alpha}$$
and
$$D^{\alpha}C = 0, \text{ for a constant } C. \tag{2.1}$$

2.1. **The Brenstein Polynomials.** The analytic form of Brenstein polynomials of degree n defined on $[0,1]$ is given by the following relation

$$B_{(i,n)}(t) = \sum_{k=0}^{n-i} \Omega_{(i,k,n)} t^{i+k} \quad i = 0, 1 \cdots n, \tag{2.2}$$

where $\Omega_{(i,k,n)} = (-1)^{k} \begin{pmatrix} n \\ i \end{pmatrix} \begin{pmatrix} n-i \\ k \end{pmatrix}$.

2.1.1. *Function Approximation.* In a Hilbert space $H = L^2[0,1]$ with the inner product

$$< f, g >= \int_0^1 f(x)g(x)dx, \tag{2.3}$$

take $Y = Span\{B_{(0,n)}(t), B_{(1,n)}(t), \cdots, B_{(n,n)}(t)\}$ a finite dimensional closed subspace. Then, an arbitrary element $f \in H$, has a unique best approximation y_0 in Y, that is

$$\exists y_0 \in Y \quad s.t \quad \forall y \in Y \quad \|f - y_o\| \leq \|f - y\|,$$

which implies that we can approximate any function in H in the form of series of Brenstein polynomials as under

$$U(t) = \sum_{i=0}^{n} c_i B_{(i,n)}(t). \tag{2.4}$$

Multiplying (2.4) by $B_{(j,n)}(t), \quad j = 0, 1 \cdots n$ and integrating from 0 to 1, we obtain set of $n+1$ algebraic equations with $n+1$ unknowns. These $n+1$ equations can be written in matrix form as

$$\begin{bmatrix} d_0 & d_1 & \cdots & d_n \end{bmatrix} = \begin{bmatrix} c_0 & c_1 & \cdots & c_n \end{bmatrix} \begin{bmatrix} \mho_{0,0} & \mho_{0,1} & \cdots & \mho_{0,n} \\ \mho_{1,0} & \mho_{1,1} & \cdots & \mho_{1,n} \\ \vdots & \vdots & \ddots & \vdots \\ \mho_{n,0} & \mho_{n,1} & \cdots & \mho_{n,n} \end{bmatrix} \tag{2.5}$$

where $d_i = \int_0^1 U(t)B_{(i,n)}(t)dt$ and $\mho_{i,j} = \int_0^1 B_{(i,n)}(t)B_{(j,n)}(t)dt$. In simplified form, we can write the above equations as

$$X_N = C_N Z_{N\times N}, \tag{2.6}$$

where $N = n + 1$ is the scale level of the scheme. From equation (2.6), we can easily calculate

$$C_N = X_N Z^{-1}_{N\times N}. \tag{2.7}$$

The matrix $Z_{N\times N}$ is called the dual matrix of the Brenstein polynomials. Once we obtain the value of c_i, we write (2.4) in matrix form as

$$U(t) = C_N B^T_N(t), \tag{2.8}$$

where C_N is the coefficients vector and $B^T_N(t)$ is the function vector of Brenstein polynomials defined as

$$B^T_N(t) = \begin{bmatrix} B_{(0,n)}(t) & B_{(1,n)}(t) & \cdots & B_{(n,n)}(t) \end{bmatrix}^T. \tag{2.9}$$

3. Main Result: Operational Matrices of fractional derivatives and integrals

In this section, we develop operational matrices of fractional order derivatives and integrals. The following theorems play basic role in our investigation.

Theorem 3.1. *Let $B^T_N(t)$ be the function vector as defined in (2.9), then the fractional order integral of order α of $B^T_N(t)$ is given by*

$$I^\alpha B^T_N(t) = P^\alpha_{N\times N} B^T_N(t), \tag{3.1}$$

where $P^\alpha_{N\times N}$ is the operational matrix of fractional integration and is defined by

$$P^\alpha_{N\times N} = \hat{P}^\alpha_{N\times N} Z^{-1}, \tag{3.2}$$

Z is the dual matrix as in (2.5) and

$$\hat{P}^\alpha_{N\times N} = \begin{bmatrix} \Theta_{0,0} & \Theta_{0,1} & \cdots & \Theta_{0,j} & \cdots & \Theta_{0,n} \\ \Theta_{1,0} & \Theta_{1,1} & \cdots & \Theta_{1,j} & \cdots & \Theta_{1,n} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \Theta_{i,0} & \Theta_{i,1} & \cdots & \Theta_{i,j} & \cdots & \Theta_{i,n} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \Theta_{n,0} & \Theta_{n,1} & \cdots & \Theta_{n,j} & \cdots & \Theta_{n,n} \end{bmatrix}, \tag{3.3}$$

where

$$\Theta_{i,j} = \sum_{k=0}^{n-i}\sum_{l=0}^{n-j} \Omega_{(i,k,n)}\Omega_{(j,l,n)} \frac{\Gamma(i+k+1)}{\Gamma(i+k+\alpha+1)(i+j+k+l+\alpha+1)}.$$

Proof. Consider a general element of the function vector and apply fractional integral of order α

$$I^\alpha B_{(i,n)} = \sum_{k=0}^{n-i} \Omega_{(i,k,n)} I^\alpha t^{i+k} = \sum_{k=0}^{n-i} \Omega_{(i,k,n)} \frac{\Gamma(i+k+1)}{\Gamma(i+k+\alpha+1)} t^{i+k+\alpha}.$$

Approximate the left side of the above equation, we have

$$\sum_{k=0}^{n-i} \Omega_{(i,k,n)} \frac{\Gamma(i+k+1)}{\Gamma(i+k+\alpha+1)} t^{i+k+\alpha} = C_N^{(i)} B_N^T(t),$$

where the vector $C_N^{(i)}$ can be obtained using the relations (2.6) and (2.7), that is,

$$C_N^{(i)} = X_N^{(i)} Z^{-1},$$

where the entries of the vector $X_N^{(i)}$ are given by

$$X_N^{(i)}(j) = \int_0^1 \sum_{k=0}^{n-i} \Omega_{(i,k,n)} \frac{\Gamma(i+k+1)}{\Gamma(i+k+\alpha+1)} t^{i+k+\alpha} B_{(j,n)}(t)dt,$$

which implies that

$$X_N^{(i)}(j) = \sum_{k=0}^{n-i}\sum_{l=0}^{n-j} \Omega_{(i,k,n)}\Omega_{(j,l,n)} \frac{\Gamma(i+k+1)}{\Gamma(i+k+\alpha+1)(i+j+k+l+\alpha+1)}. \tag{3.4}$$

Evaluating (3.4) for $i = 0, 1, \cdots n$ and $i = 0, 1, \cdots n$, and writing in the matrix form, we get

$$\begin{bmatrix} I^\alpha B_{(0,n)} \\ I^\alpha B_{(1,n)} \\ \vdots \\ I^\alpha B_{(i,n)} \\ \vdots \\ I^\alpha B_{(n,n)} \end{bmatrix} = \begin{bmatrix} X_N^{(0)} Z^{-1} B_N^T(t) \\ X_N^{(1)} Z^{-1} B_N^T(t) \\ \vdots \\ X_N^{(i)} Z^{-1} B_N^T(t) \\ \vdots \\ X_N^{(n)} Z^{-1} B_N^T(t) \end{bmatrix}.$$

Using the notation

$$\Theta_{i,j} = \sum_{k=0}^{n-i}\sum_{l=0}^{n-j}\Omega_{(i,k,n)}\Omega_{(j,l,n)}\frac{\Gamma(i+k+1)}{\Gamma(i+k+\alpha+1)(i+j+k+l+\alpha+1)}$$

we have the desired result

$$I^{\alpha}B_N^T(t) = \hat{P}^{\alpha}_{N\times N}Z^{-1}B_N^T(t) = P^{\alpha}_{N\times N}B_N^T(t),$$

where $\hat{P}^{\alpha}_{N\times N}Z^{-1} = P^{\alpha}_{N\times N}$. □

Theorem 3.2. *Let $B_N^T(t)$ be the function vector as defined in (2.9) then the fractional order derivative of order α of $B_N^T(t)$ is given by*

$$D^{\alpha}B_N^T(t) = G^{\alpha}_{N\times N}B_N^T(t), \tag{3.5}$$

where $G^{\alpha}_{N\times N}$ is the operational matrix of fractional order derivative and is defined by

$$G^{\alpha}_{N\times N} = \hat{G}^{\alpha}_{N\times N}Z^{-1}, \tag{3.6}$$

where

$$\hat{G}^{\alpha}_{N\times N} = \begin{bmatrix} \Theta_{0,0} & \Theta_{0,1} & \cdots & \Theta_{0,j} & \cdots & \Theta_{0,n} \\ \Theta_{1,0} & \Theta_{1,1} & \cdots & \Theta_{1,j} & \cdots & \Theta_{1,n} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \Theta_{i,0} & \Theta_{i,1} & \cdots & \Theta_{i,j} & \cdots & \Theta_{i,n} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \Theta_{n,0} & \Theta_{n,1} & \cdots & \Theta_{n,j} & \cdots & \Theta_{n,n} \end{bmatrix}, \tag{3.7}$$

and $\Theta_{i,j}$ is defined for two different cases

Case 1:$(i < \lceil\alpha\rceil)$

$$\Theta_{i,j} = \sum_{k=\lceil\alpha\rceil}^{n-i}\sum_{l=0}^{n-j}\Omega_{(i,k,n)}\Omega_{(j,l,n)}\frac{\Gamma(i+k+1)}{\Gamma(i+k-\alpha+1)(i+j+k+l-\alpha+1)}. \tag{3.8}$$

Case 2:$(i \geq \lceil\alpha\rceil)$

$$\Theta_{i,j} = \sum_{k=0}^{n-i}\sum_{l=0}^{n-j}\Omega_{(i,k,n)}\Omega_{(j,l,n)}\frac{\Gamma(i+k+1)}{\Gamma(i+k-\alpha+1)(i+j+k+l-\alpha+1)}. \tag{3.9}$$

Proof. Consider a general element in the function vector and apply fractional order derivative of order α, we have

$$D^{\alpha}B_{(i,n)} = \sum_{k=0}^{n-i}\Omega_{(i,k,n)}D^{\alpha}t^{i+k}. \tag{3.10}$$

It should be noted that in the polynomial function $B_{(i,n)}$, the power of the variable t is in ascending order and the lowest power is i. We discuss two cases

Case 1:$(i < \lceil \alpha \rceil)$

Using the definition of fractional integration we get

$$D^\alpha B_{(i,n)} = \sum_{k=\lceil \alpha \rceil}^{n-i} \Omega_{(i,k,n)} \frac{\Gamma(i+k+1)}{\Gamma(i+k-\alpha+1)} t^{i+k-\alpha}. \tag{3.11}$$

As in the above lemma, we can easily approximate the left side of the above equation as

$$\sum_{k=\lceil \alpha \rceil}^{n-i} \Omega_{(i,k,n)} \frac{\Gamma(i+k+1)}{\Gamma(i+k-\alpha+1)} t^{i+k-\alpha} = C_N^{(i)} B_N^T(t), \tag{3.12}$$

where the vector $C_N^{(i)}$ is given by

$$C_N^{(i)} = X_N^{(i)} Z^{-1}, \tag{3.13}$$

and the entries of the vector $X_N^{(i)}$ are given by

$$\begin{aligned} X_N^{(i)}(j) &= \int_0^1 \sum_{k=\lceil \alpha \rceil}^{n-i} \Omega_{(i,k,n)} \frac{\Gamma(i+k+1)}{\Gamma(i+k-\alpha+1)} t^{i+k-\alpha} B_{(j,n)}(t) dt \\ &= \sum_{k=\lceil \alpha \rceil}^{n-i} \sum_{l=0}^{n-j} \Omega_{(i,k,n)} \Omega_{(j,l,n)} \frac{\Gamma(i+k+1)}{\Gamma(i+k-\alpha+1)(i+j+k+l-\alpha+1)} \end{aligned} \tag{3.14}$$

Case 2:$(i \geq \lceil \alpha \rceil)$ If $i \geq \lceil \alpha \rceil$, then using the above procedure we can easily write

$$X_N^{(i)}(j) = \sum_{k=0}^{n-i} \sum_{l=0}^{n-j} \Omega_{(i,k,n)} \Omega_{(j,l,n)} \frac{\Gamma(i+k+1)}{\Gamma(i+k-\alpha+1)(i+j+k+l-\alpha+1)}, \tag{3.15}$$

Evaluating this result for $i = 0, 1, \cdots n$ and $i = 0, 1, \cdots n$, taking care of the both cases and writing in the matrix form we get

$$\begin{bmatrix} D^\alpha B_{(0,n)} \\ D^\alpha B_{(1,n)} \\ \vdots \\ D^\alpha B_{(i,n)} \\ \vdots \\ D^\alpha B_{(n,n)} \end{bmatrix} = \begin{bmatrix} X_N^{(0)} Z^{-1} B_N^T(t) \\ X_N^{(1)} Z^{-1} B_N^T(t) \\ \vdots \\ X_N^{(i)} Z^{-1} B_N^T(t) \\ \vdots \\ X_N^{(n)} Z^{-1} B_N^T(t) \end{bmatrix}. \tag{3.16}$$

On further simplification and representing

$$\Theta_{i,j} = \sum_{k=\lceil\alpha\rceil}^{n-i} \sum_{l=0}^{n-j} \Omega_{(i,k,n)} \Omega_{(j,l,n)} \frac{\Gamma(i+k+1)}{\Gamma(i+k-\alpha+1)(i+j+k+l-\alpha+1)} \qquad i < \lceil\alpha\rceil$$

$$\Theta_{i,j} = \sum_{k=0}^{n-i} \sum_{l=0}^{n-j} \Omega_{(i,k,n)} \Omega_{(j,l,n)} \frac{\Gamma(i+k+1)}{\Gamma(i+k-\alpha+1)(i+j+k+l-\alpha+1)} \qquad i \geq \lceil\alpha\rceil$$

we get

$$D^\alpha B_N^T(t) = \hat{G}_{N\times N}^\alpha Z^{-1} B_N^T(t). \tag{3.17}$$

Using the notation $\hat{G}_{N\times N}^\alpha Z^{-1} = G_{N\times N}^\alpha$, we get desire results. □

4. Application of the new Matrices

The new operational matrices play a very important role for solutions of fractional differential equations and coupled systems of fractional differential equations.

4.1. **Fractional differential equations.** Consider the following generalized class of FDEs

$$D^\alpha U(t) = \sum_{i=0}^{n} d_i D^{\beta_i} U(t) + g(t), \tag{4.1}$$

subject to initial conditions

$$U^i(0) = u_i,\ i = 0, 1, \cdots n,$$

where u_i are all real constant, $n < \alpha \leq n+1$, $t \in [0, 1]$, $U(t)$ is the unknown solution to be determined, $g(t)$ is the given source term and d_i for $i = 0, 1 \cdots n$ are real constants. We seek the solution of the above problem in term of Brenstein polynomials such that

$$D^\alpha U(t) = K_N B_N^T(t). \tag{4.2}$$

Applying fractional integral of order α and by using the given initial conditions and making use of Theorem 3.1, we write

$$U(t) - \sum_{j=0}^{n} t^j u_j = K_N P^{\alpha}_{N\times N} B^T_N(t), \tag{4.3}$$

which implies that

$$U(t) = K_N P^{\alpha}_{N\times N} B^T_N(t) + F^1_N B^T_N(t) = \hat{K}_N B^T_N(t), \tag{4.4}$$

where $F^1_N B^T_N(t) = \sum_{j=0}^{n} t^j u_j$ and

$$\hat{K}_N = K_N P^{\alpha}_{N\times N} + F^1_N. \tag{4.5}$$

Using (4.4) and Theorem 3.2, we write

$$D^{\beta_i} U(t) = \hat{K}_N G^{\beta_i}_{N\times N} B^T_N(t). \tag{4.6}$$

Approximating $g(t) = F^2_N B^T_N(t)$ and using (4.6) in (4.1) we get

$$K_N B^T_N(t) = \sum_{i=0}^{n} d_i \hat{K}_N G^{\beta_i}_{N\times N} B^T_N(t) + F^2_N B^T_N(t). \tag{4.7}$$

which implies that

$$\{K_N - \hat{K}_N \sum_{i=0}^{n} d_i G^{\beta_i}_{N\times N} - F^2_N\} B^T_N(t) = 0. \tag{4.8}$$

Canceling out the common term we can write

$$K_N - \hat{K}_N \sum_{i=0}^{n} d_i G^{\beta_i}_{N\times N} - F^2_N = 0. \tag{4.9}$$

Now using (4.5) we can write the above equation as

$$K_N - K_N P^{\alpha}_{N\times N} \sum_{i=0}^{n} d_i G^{\beta_i}_{N\times N} - F^1_N \sum_{i=0}^{n} d_i G^{\beta_i}_{N\times N} - F^2_N = 0. \tag{4.10}$$

Equation (4.10) is easily solvable algebric equation and can be easily solved for the unknown coefficient vector K_N. Using the value of K_N in equation (4.4) will lead us to the approximate solution of the problem.

4.2. **Coupled system of FDEs.** Consider the following class of coupled system of FDEs.

$$\begin{aligned} D^{\alpha_1}U(t) &= \sum_{i=0}^{n} a_i D^{\beta_{1i}}U(t) + \sum_{i=0}^{n} b_i D^{\beta_{2i}}V(t) + g_1(t), \\ D^{\alpha_2}V(t) &= \sum_{i=0}^{n} c_i D^{\gamma_{1i}}U(t) + \sum_{i=0}^{n} d_i D^{\gamma_{2i}}V(t) + g_2(t), \end{aligned} \tag{4.11}$$

subject to the initial conditions

$$U^i(0) = u_i, \qquad V^i(0) = v_i, \qquad i = 0, 1 \cdots n, \tag{4.12}$$

where a_i, b_i, c_i, d_i, u_i and v_i are real constants and $U(t)$ and $V(t)$ are unknown solutions to be determined, $g_1(t)$ and $g_2(t)$ are the known source terms to the system. Order of derivatives are analogously defined as in previous problem.

We seek the solutions to the system in form of Brenstien polynomials such that

$$D^{\alpha_1}U(t) = K_N B_N^T(t),\ D^{\alpha_2}V(t) = L_N B_N^T(t). \tag{4.13}$$

By the application of fractional order integral of order α_1 and α_2, making use of Theorem 3.1 and after simplifying as in the above section, we get

$$U(t) = \hat{K}_N B_N^T(t),\ V(t) = \hat{L}_N B_N^T(t), \tag{4.14}$$

where $\hat{K}_N = K_N P^{\alpha_1}_{N\times N} + F_N^1$ and $\hat{L}_N = L_N P^{\alpha_2}_{N\times N} + F_N^2$, where $F_N^1 B_N^T(t) = \sum_{i=0}^n u_i t^i$ and $F_N^2 B_N^T(t) = \sum_{i=0}^n v_i t^i$. Now using Theorem 3.2 and (4.14), we get the following estimates

$$\begin{aligned} & a_i D^{\beta_{1i}}U(t) = a_i \hat{K}_N G^{\beta_{1i}}_{N\times N} B_N^T(t),\ b_i D^{\beta_{2i}}V(t) = b_i \hat{L}_N G^{\beta_{2i}}_{N\times N} B_N^T(t), \\ & c_i D^{\gamma_{1i}}U(t) = c_i \hat{K}_N G^{\gamma_{1i}}_{N\times N} B_N^T(t),\ d_i D^{\gamma_{2i}}V(t) = d_i \hat{L}_N G^{\gamma_{2i}}_{N\times N} B_N^T(t), \\ & g_1(t) = F_N^3 B_N^T(t) \text{ and } g_2(t) = F_N^4 B_N^T(t). \end{aligned} \tag{4.15}$$

Know using the estimates (4.15) and (4.13) in the main (4.11) and writing in matrix form we get

$$\begin{aligned} \begin{bmatrix} K_N B_N^T(t) \\ L_N B_N^T(t) \end{bmatrix} &= \begin{bmatrix} \sum_{i=0}^n a_i \hat{K}_N G^{\beta_{1i}}_{N\times N} B_N^T(t) \\ \sum_{i=0}^n d_i \hat{L}_N G^{\gamma_{2i}}_{N\times N} B_N^T(t) \end{bmatrix} \\ &\quad + \begin{bmatrix} \sum_{i=0}^n b_i \hat{L}_N G^{\beta_{2i}}_{N\times N} B_N^T(t) \\ \sum_{i=0}^n c_i \hat{K}_N G^{\gamma_{1i}}_{N\times N} B_N^T(t) \end{bmatrix} + \begin{bmatrix} F_N^3 B_N^T(t) \\ F_N^4 B_N^T(t) \end{bmatrix}. \end{aligned} \tag{4.16}$$

Taking the transpose of the above matrix equation, using simplified notation and after a short simplification we can write

$$\begin{aligned} \left[\begin{array}{cc} K_N & L_N \end{array}\right] \widehat{B} &= \left[\begin{array}{cc} \hat{K}_N & \hat{L}_N \end{array}\right] \left[\begin{array}{cc} \sum_{i=0}^{n} a_i G_{N\times N}^{\beta_{1i}} & O_{N\times N} \\ O_{N\times N} & \sum_{i=0}^{n} d_i G_{N\times N}^{\gamma_{2i}} \end{array}\right] \widehat{B} \\ &+ \left[\begin{array}{cc} \hat{K}_N & \hat{L}_N \end{array}\right] \left[\begin{array}{cc} O_{N\times N} & \sum_{i=0}^{n} b_i G_{N\times N}^{\beta_{2i}} \\ \sum_{i=0}^{n} c_i G_{N\times N}^{\gamma_{1i}} & O_{N\times N} \end{array}\right] \widehat{B} + \left[\begin{array}{cc} F_N^3 & F_N^4 \end{array}\right] \widehat{B}, \end{aligned} \tag{4.17}$$

where the matrix $\widehat{B} = \left[\begin{array}{cc} B_N^T(t) & O_N \\ O_N & B_N^T(t) \end{array}\right]$. Canceling out the common factor and simplifying we get

$$\begin{aligned} &\left[\begin{array}{cc} K_N & L_N \end{array}\right] - \left[\begin{array}{cc} \hat{K}_N & \hat{L}_N \end{array}\right] \left[\begin{array}{cc} \sum_{i=0}^{n} a_i G_{N\times N}^{\beta_{1i}} & \sum_{i=0}^{n} b_i G_{N\times N}^{\beta_{2i}} \\ \sum_{i=0}^{n} c_i G_{N\times N}^{\gamma_{1i}} & \sum_{i=0}^{n} d_i G_{N\times N}^{\gamma_{2i}} \end{array}\right] \\ &- \left[\begin{array}{cc} F_N^3 & F_N^4 \end{array}\right] = 0. \end{aligned} \tag{4.18}$$

Using the values of $\hat{K}$ and $\hat{L}$ in the above equation we get

$$\begin{aligned} &\left[\begin{array}{cc} K_N & L_N \end{array}\right] - \left[\begin{array}{cc} K_N & L_N \end{array}\right] \left[\begin{array}{cc} P_{N\times N}^{\alpha_1} \sum_{i=0}^{n} a_i G_{N\times N}^{\beta_{1i}} & P_{N\times N}^{\alpha_1} \sum_{i=0}^{n} b_i G_{N\times N}^{\beta_{2i}} \\ P_{N\times N}^{\alpha_2} \sum_{i=0}^{n} c_i G_{N\times N}^{\gamma_{1i}} & P_{N\times N}^{\alpha_2} \sum_{i=0}^{n} d_i G_{N\times N}^{\gamma_{2i}} \end{array}\right] \\ &- \left[\begin{array}{cc} F_N^1 & F_N^2 \end{array}\right] \left[\begin{array}{cc} \sum_{i=0}^{n} a_i G_{N\times N}^{\beta_{1i}} & \sum_{i=0}^{n} b_i G_{N\times N}^{\beta_{2i}} \\ \sum_{i=0}^{n} c_i G_{N\times N}^{\gamma_{1i}} & \sum_{i=0}^{n} d_i G_{N\times N}^{\gamma_{2i}} \end{array}\right] - \left[\begin{array}{cc} F_N^3 & F_N^4 \end{array}\right] = 0, \end{aligned} \tag{4.19}$$

which is easily solvable system of algebraic equation and can be easily solved for the unknown K_N and L_N. Using the values of K_N and L_N in equation (4.14) we can get the approximate solution of the problem.

5. Test Problems

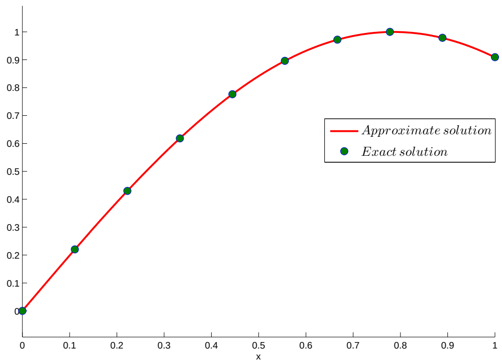
We solve some test problems with the new technique. We observe that the method is highly efficient and provide a very good approximations to the solutions.

Example 1: Consider the following fractional differential equation

$$D^\alpha U(t) = 2DU(t) + 3U(t) - 4cos(2t) - 7sin(2t),$$

with initial condition

FIGURE 1. The comparison of exact and approximate solution of Example 1 at N=5.



$$U(0) = 0 \qquad U'(0) = 2.$$

Note that the order of derivative is $1 < \alpha \leq 2$. The exact solution of this problem at $\alpha = 2$ is $U(t) = sin(2t)$ however the exact solution at fractional value of α is not known. It is one of the well known property of the fractional differential equations that the solution of fractional differential equations at fractional values approaches to the solution at the integer value as the order of derivative approaches from fractional value to the corresponding integer value. We use this property to check the accuracy of the solution at fractional values. At first we fix $\alpha = 2$ and approximate the solution with different scale level. We observe that the solution becomes more and more accurate as we increase the scale level. Figure 1 shows the comparison of the exact and approximate solution at $N = 5$. It can be easily observed that the approximate solutions matches very well with the exact solution of the problem.We approximate error of the scheme at different scale level and found that as the scale level increases, the absolute error decreases. This phenomena is visualized in Figure 2. We also approximate solution of this problem at fractional value of α and observe that as $\alpha \to 2$ the approximate solution approaches to the exact solution see Figure 3. We observe that the scheme provides a much more accurate estimate of the solution. To show the convergence of the scheme we approximate the quantity $\int_0^1 |U_{exact} - U_{approx}|dt$ for different value of N, and found that the norm of error decreases with a high speed with the increase of scale level N as demonstrated in Figure 4.

FIGURE 2. The absolute error of Example 1 at different value of N, setting $\alpha = 1$.

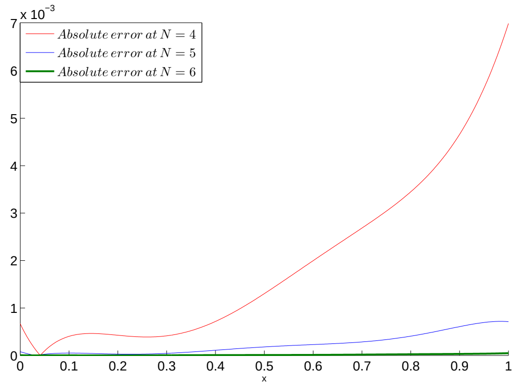
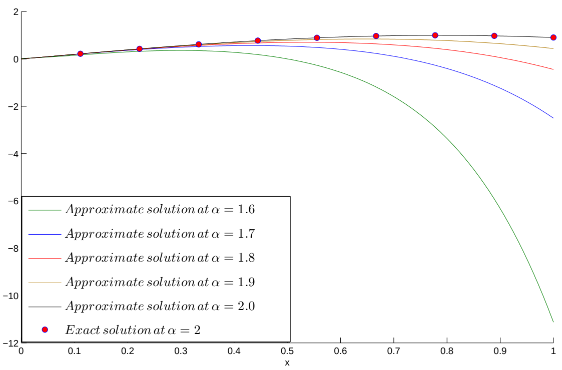


FIGURE 3. The approximate solution of Example 1 at fractional value of α, N=5.

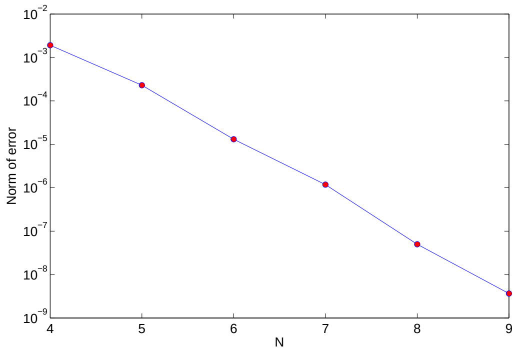


Example 2(Two tank mixing problem): Consider the governing equations of two tank mixing problem

$$D^{\alpha}U(t) = -0.02(U(t)) + 0.02(V(t)), \tag{5.1}$$

$$D^{\alpha}V(t) = 0.02(U(t)) - 0.02(V(t)), \tag{5.2}$$

FIGURE 4. The norm of error of Example 1 .



with initial conditions $U(0) = -0.5$ and $V(0) = 1.5$. The exact solution for $\alpha = 1$ is

$$U(t) = 0.5 - e^{-0.04t},$$

and

$$V(t) = 0.5 + e^{-0.04t}.$$

where $0 < \alpha \leq 1$. We fix $\alpha = 1$ and approximate the solutions of the problem with this new technique. We observe that the new method provides a very high accurate approximate solution. We approximate the solutions of this problem with different scale level. We observe that increase in the scale level results in the accuracy of approximate solution. Figure 5 shows the comparison of exact and approximate solutions. We see that the exact and approximate solution marches very well. We also approximate the solution at some fractional value of α, see Figure 6. We observe that the solution approaches to the exact solution at $\alpha = 1$ as the value of α approaches 1. We approximate absolute error at different value of N. We observe that the absolute error is less that 10^{-15}, which is very high accuracy for such types of complicated problems. Figure 7 shows this phenomena.

Example 3(Fractionally damped coupled spring mass system [27]): Consider the equations of fractionally damped coupled system of two masses. The governing equations are given as

$$m_1\, D^{\alpha}U(t) = -(c_1+c_2)\frac{dU(t)}{dt} - (k_1+k_2)U(t) + c_2\frac{dV(t)}{dt} + k_2V(t) + F_1(t), \quad (5.3)$$

FIGURE 5. The comparison of exact and approximate solution of Example 2 at N=5. Fig (a) shows U(t) while Fig (b) shows V(t).

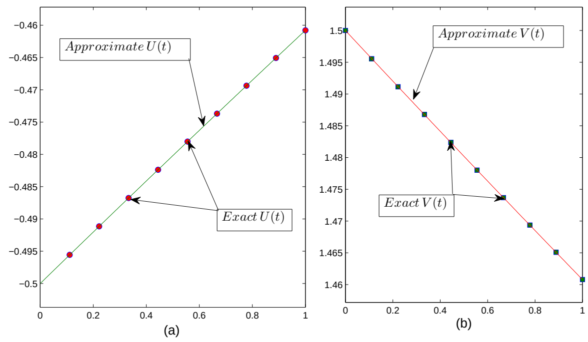
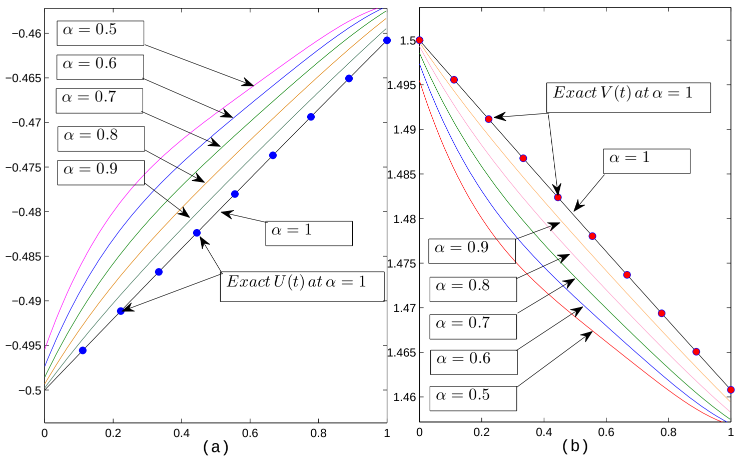


FIGURE 6. The approximate solutions of Example 2 at fractional value of α. Fig (a) shows U(t) while Fig (b) shows V(t).

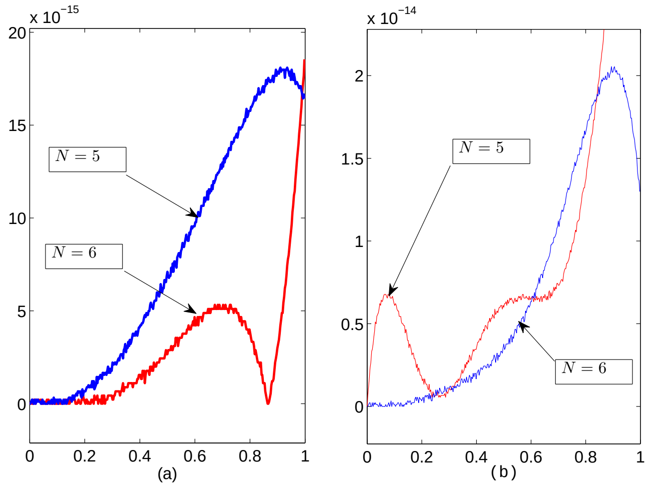


$$m_2\ D^{\alpha}V(t) = c_2\frac{dU(t)}{dt} + k_2U(t) - c_2\frac{dV(t)}{dt} - k_2V(t) + F_2(t), \tag{5.4}$$

where $1 \le \alpha \le 2$, c_1, c_2, c_3 are the damping parameter k_1, k_2, k_3 are the spring constant, with

$$F_1(t) = \frac{301\,\sin(\pi\,t)}{200} - \frac{99\,\cos(\pi\,t)}{100} + \frac{259\,\pi\,\cos(\pi\,t)}{100} + \frac{351\,\pi\,\sin(\pi\,t)}{200} - \frac{7\,\pi^2\,\sin(\pi\,t)}{10},$$

FIGURE 7. The absolute error of Example 2 at different scale level. Fig (a) shows U(t) while Fig (b) shows V(t).



and

$$F_2(t) = \frac{513\,\cos(\pi\,t)}{200} - \frac{77\,\sin(\pi\,t)}{100} - \frac{273\,\pi\,\cos(\pi\,t)}{200} - \frac{153\,\pi\,\sin(\pi\,t)}{50} - \frac{9\,\pi^2\,\cos(\pi\,t)}{10}.$$

For $\alpha = 2, m_1 = 1, m_2 = 1, c_1 = 1.75, c_2 = 1.95, c_3 = 1.45, k_1 = 1.05, k_2 = 1.1$ *and* $k_3 = 1.75$ along with the initial condition

$$V(0) = 9/10 \textit{ and } V'(0) = 0,\; U(0) = 0 \textit{ and } U'(0) = 7\pi/10$$

the exact solution to the problem is

$$U(t) = \frac{7\,\sin(\pi\,t)}{10},$$

and

$$V(t) = \frac{9\,\cos(\pi\,t)}{10}.$$

We approximate the solution of this problem with this new method. We observe that as expected the method provides a very good approximation to the solution of the problem. At first we approximate the solutions of the problem at $\alpha = 2$ (because the exact solution at $\alpha = 2$ is known). We observe that at much small scale level the method provides a very good approximation to the solution, as can be seen from Figure 8 the exact and approximate solutions are equal. We approximate the absolute

FIGURE 8. The comparison of exact and approximate solution of example 3 at N=5.

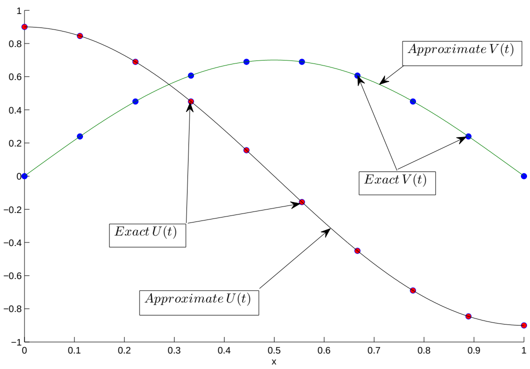
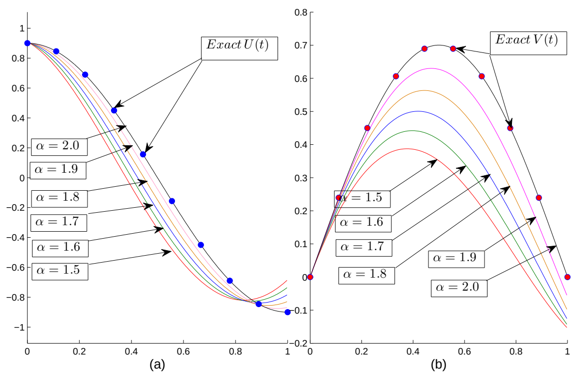


FIGURE 9. The approximate solutions of example 3 at fractional value of α. Fig (a) shows U(t) while Fig (b) shows V(t).

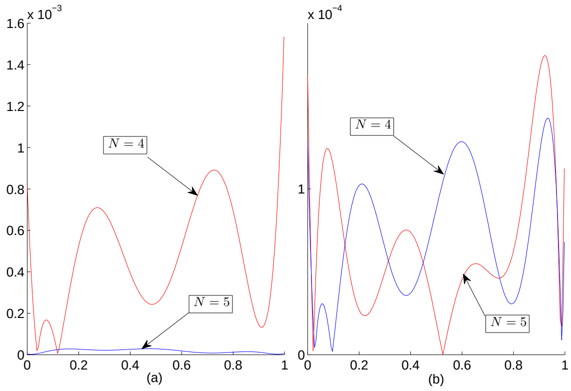


error at different scale level of N using formula

$$U_{error} = |U_{exact} - U_{approx}|.$$

And observe that the absolute error is much more less than 10^{-4}, see Figure 10. We also approximate the solution at some fractional value of α and observe that as $\alpha \to 2$ the approximate solutions approaches to the exact solutions at $\alpha = 2$. Which guarantees the accuracy of solution at fractional value of α. Figure 9 shows this phenomena. In Figure 8 and Figure 9 the subplot (a) represents the approximation of $U(t)$ and subplot (b) represents the approximation of $V(t)$.

FIGURE 10. The absolute error of example 2 at different scale level.Fig (a) shows U(t) while Fig (b) shows V(t).



6. Conclusion

From above analysis and observation we conclude that the new method is very efficient for the solution of fractional differential equations including coupled system. The method can be easily extended to solve such problems with other kinds of boundary condition. As well as the new operational matrices can be easily extended to two dimensional case which will help in the solution of fractional order partial differential equation. Our future work is related to extension of the method to solve fractional order partial differential equations.

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