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Monodromy problem for the degenerate critical points

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Abstract For the polynomial planar vector fields with a hyperbolic or nilpotent critical point at the origin, the monodromy problem has been solved, but for the strongly degenerate critical points this problem is still open. When the critical point is monodromic, the stability problem or the center- focus problem is an open problem too. In this paper we will consider the polynomial planar vector fields with a degenerate critical point at the origin. At first we give some normal form for the systems which has no characteristic directions. Then we consider the systems with some characteristic directions at which the origin is still a monodromic critical point and we give a monodromy criterion. Finally we clarify our work by some examples.

Keywords. Monodromy problem, center-focus problem, degenerate critical point, hyperbolic critical point, nilpotent critical point, blow up method.

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1. Introduction

Consider the planar system

$$\begin{cases} \dot{x} &= P(x,y) \\ \dot{y} &= Q(x,y), \end{cases} \tag{1.1}$$

where P and Q are analytic functions in some planar region Ω. We say that (x_0, y_0) is a singular point or critical point of the system (1.1), if $P(x_0, y_0) = Q(x_0, y_0) = 0$. A critical point p of the system (1.1) is called monodromic if there is no solution of the system tending or leaving p with a concrete slope and all the orbits turned around it. The monodromy problem is determining whether the critical point is monodromic

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or not. A critical point p is called focus if every trajectory started in a neighborhood of p, spirals around p and p is called center if it is surrounded by closed orbits. In analytic systems a monodromic critical point is either a focus or a center. This fact has been proved as a consequence of the finiteness of the limit cycles at the beginning of the 19th century [6].

When both the eigenvalues of the Jacobian matrix at the critical point differs from zero, the critical point is called non-degenerate. In this case the monodromy problem has been solved. Indeed, when the eigenvalues of the Jacobian matrix at the critical point are complex numbers with nonzero imaginary parts, the critical point is monodromic. If the real part of these eigenvalues are different from zero, i.e. the critical point is hyperbolic, then the critical point is a focus or center, while if their real part are zero, then the critical point is a center [7].

When only one of the eigenvalues of the Jacobian matrix at the critical point is zero, then the critical point is not monodromic [11]. When both the eigenvalues of the Jacobian matrix at the critical point equals zero but the Jacobian matrix is not identically zero, the critical point is called nilpotent, and when the planar system has no linear part the critical point is called strongly degenerate (in short degenerate) or linearly zero.

As soon as the critical point is monodromic, the stability problem or the center-focus problem is an open problem too. For the hyperbolic critical point the center-focus problem has been solved theoretically by Poincaré [12] and Lyapunov [8], by means of the so called Lyapunov constants. For the nilpotent critical point the monodromy problem was solved by Andreev [3] and the center- focus problem was solved by Moussu [10].

But for the strongly degenerate points almost nothing has been solved and it has been rarely studied. For this type of critical point even the monodromy problem is very complicated, see for instance [5, 9]. The main technique to investigate the strongly degenerate critical points is the blowing up method [1]. In this method by means of a non-diffeomorphism change of variable, one can explode the critical point to a line or a circle. Considering the critical points of the new system on this line or circle is useful to study the original critical points. In the new system if some of the critical points are degenerate, then the process is repeated. By Domortier Theorem this iterative process of desingularization is finite [4].

In this paper we will consider the degenerate critical points of the planar polynomial systems. Without loose of generality and by a linear transformation, we suppose that the critical point is moved to the origin.

In Section 2, we give a normal form for the systems with no characteristic directions at which the origin is a monodromic critical point. In Section 3, by means of the polar blow up method we give a monodromy criterion. Finally in Section 4 we give some examples to clarify our results.

2. Systems with no characteristic directions

Consider the system of differential equations

$$\begin{cases} \dot{x} &= P(x,y), \\ \dot{y} &= Q(x,y), \end{cases} \tag{2.1}$$

where

$$P(x,y) = P_m(x,y) + \sum_{i=1}^{n} P_{m+i}(x,y),$$

$$Q(x,y) = Q_m(x,y) + \sum_{i=1}^{n} Q_{m+i}(x,y),$$

P_k, Q_k are homogeneous polynomials of degree k, and at least one of the P_m, Q_m is not identically zero.

Suppose that $m > 1$, hence the origin is a degenerate critical point of (2.1). In the polar coordinates and after the change of variable $ds = dt\, r^{m-1}$ and denoting the derivative with respect to s by $\frac{dr}{ds} = \dot{r}$, $\frac{d\theta}{ds} = \dot{\theta}$, one can show easily that the system (2.1) becomes

$$\begin{cases} \dot{r} &= \mathcal{R}(\theta)r + \mathcal{O}(r^2), \\ \dot{\theta} &= \mathcal{F}(\theta) + \mathcal{O}(r), \end{cases} \tag{2.2}$$

where

$$\mathcal{R}(\theta) = \cos\theta\, P_m(\cos\theta, \sin\theta) + \sin\theta\, Q_m(\cos\theta, \sin\theta),$$

$$\mathcal{F}(\theta) = \cos\theta\, Q_m(\cos\theta, \sin\theta) - \sin\theta\, P_m(\cos\theta, \sin\theta).$$

If $\mathcal{F} \equiv 0$ then the origin is called a dicritical point. In this case we have $P_m = xW_{m-1}$, $Q_m = yW_{m-1}$, where W_{m-1} is a homogeneous polynomial of degree $m-1$. If ν is a root of W_{m-1}, where $\nu = \tan(\theta^*)$, then θ^* is called a singular direction [1]. In this case for every nonsingular direction θ there exists exactly one semipath tending to the origin which is tangent to θ in forward or backward time and for a singular direction θ^*, there might be be either no semipaths tending to the origin tangent to θ^*, or a finite number, or infinitely many [1]. In particular a dicritical point is not monodromic.

If $\mathcal{F}(\theta) \not\equiv 0$, then the roots of $\mathcal{F}$ in the interval $[0, 2\pi)$ is called characteristic directions. Thus the characteristic directions are the solutions of the following equation

$$\cos\theta\, Q_m(\cos\theta, \sin\theta) - \sin\theta\, P_m(\cos\theta, \sin\theta) = 0. \tag{2.3}$$

In this case every solution of (2.1) tending or leaving the origin are tangent to one of the characteristic directions [1, 11]. Thus if there are no characteristic direction, then the origin is monodromic. Unfortunately it is possible that the system has some characteristic directions and there is no solution of the system tending or leaving the origin tangent to those characteristic directions. Thus the origin is monodromic, for instance see Example 4.2.

In this section we introduce a normal form for the planar polynomial vector fields with a degenerate critical point at the origin having no characteristic directions.

Notation 2.1. If the system (2.2) has no characteristic directions, we call the origin as a pure monodromic critical point.

The origin is called a conditional monodromic critical point, whenever there are some characteristic directions and there are no solutions of the system that tend to the origin tangent to those characteristic directions.

In the following we consider the system

$$\begin{cases} \dot{x} &= P(x,y), \\ \dot{y} &= Q(x,y), \end{cases} \tag{2.4}$$

where $P = P_m(x,y) + \sum_{i=1}^{n} P_{m+i}(x,y)$, $Q = Q_m(x,y) + \sum_{r=i}^{n} Q_{m+i}(x,y)$ and P_k, Q_k are homogeneous polynomials of degree k. Suppose that $m > 1$ and at least one of the P_m, Q_m is not identically zero.

Theorem 2.2. *If the origin is a pure monodromic critical point of the system (2.4), then m is odd and the system can be written as*

$$\begin{cases} \dot{x} &= -y^m + \bar{P}(x,y), \\ \dot{y} &= x^m + \bar{Q}(x,y), \end{cases} \tag{2.5}$$

where

$$\bar{P}(x,y) = xP_{m-1}(x,y) + \sum_{i=1}^{n} P_{m+i}(x,y),$$

$$\bar{Q}(x,y) = yQ_{m-1}(x,y) + \sum_{i=1}^{n} Q_{m+i}(x,y).$$

Lemma 2.3. *If the system (2.4) has a pure monodromic critical point at the origin, then the system can be written as*

$$\begin{cases} \dot{x} &=& ay^m + \hat{P}(x,y), \\ \dot{y} &=& bx^m + \hat{Q}(x,y), \end{cases} \tag{2.6}$$

where at least one of the a, b does not equal zero,

$$\hat{P}(x,y) = xP_{m-1}(x,y) + \sum_{i=1}^{n} P_{m+i}(x,y),$$

and

$$\hat{Q}(x,y) = yQ_{m-1}(x,y) + \sum_{i=1}^{n} Q_{m+i}(x,y).$$

Proof. Let $P_m = \sum_{j=0}^{m} \alpha_j x^j y^{m-j}$, $Q_m = \sum_{j=0}^{m} \beta_j x^j y^{m-j}$. If $\alpha_0, \beta_m \neq 0$ the proof is obvious by $a = \alpha_0, b = \beta_m$, $\hat{P}(x,y) = P(x,y) - ay^m, \hat{Q}(x,y) = Q(x,y) - bx^m$. If $\alpha_0 = \beta_m = 0$ consider $\tilde{P}(x,y) = \frac{P_m(x,y)}{x}$, $\tilde{Q}(x,y) = \frac{Q_m(x,y)}{y}$, both $\tilde{P}, \tilde{Q}$ are polynomials and Eq. (2.3) becomes

$$\cos\theta \sin\theta \tilde{Q_m}(\cos\theta, \sin\theta) - \sin\theta\cos\theta \tilde{P_m}(\cos\theta, \sin\theta) = 0.$$

This equation has at least four solutions in $[0, 2\pi)$. Thus the origin can not be a pure monodromic critical point. If $\alpha_0 = 0$ and $\beta_m \neq 0$, then Eq. (2.3) becomes

$$\cos\theta \sin\theta \tilde{Q_m}(\cos\theta, \sin\theta) - \sin\theta P_m(\cos\theta, \sin\theta) = 0.$$

This equation has at least two roots in the interval $[0, 2\pi)$. Similarly if $\alpha_0 \neq 0$ and $\beta_m = 0$, then the origin is not a pure monodromic critical point. $\square$

Lemma 2.4. *Consider the system*

$$\begin{cases} \dot{x} &=& ay^m + \hat{P}(x,y), \\ \dot{y} &=& bx^m + \hat{Q}(x,y), \end{cases} \tag{2.7}$$

where $\hat{P}, \hat{Q}$ are polynomials as in lemma 2.3. If the origin is a pure monodromic critical point, then m is odd.

Proof. Let $P_m = ay^m + \sum_{j=1}^{m} \alpha_j x^j y^{m-j}$, $Q_m = bx^m + \sum_{j=0}^{m-1} \beta_j x^j y^{m-j}$. If $m = 2p$, then Eq. (2.3) becomes

$$b\cos^{2p+1}\theta + \cos\theta\left(\sum_{j=0}^{m-1} \beta_j \cos^j\theta \sin^{m-j}\theta\right), \tag{2.8}$$

$$-a\sin^{2p+1}\theta - \sin\theta\left(\sum_{j=1}^{m} \alpha_j \cos^j\theta \sin^{m-j}\theta\right) = 0. \tag{2.9}$$

In the interval $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ we can divide the equation by $\cos^{2p+1}(\theta)$ and we have

$$-a\tan^{2p+1}(\theta) + H(\cos\theta, \sin\theta) = 0,$$

where H is an analytic function in the interval $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$. Let

$$G(\theta) := -a\tan^{2p+1}(\theta) + H(\cos\theta, \sin\theta),$$

then we have $lim_{\theta\to\frac{\pi}{2}^+}G(\theta) = \infty$ and $lim_{\theta\to\frac{3\pi}{2}^-}G(\theta) = \infty$, where the sign of ∞ at the first limit is $-sgn(a)$ and in the second one is $sgn(a)$. By the intermediate value theorem $G(\theta)$ has at least one roots in $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ and the origin can not be a pure monodromic critical point of the system. Therefore m is odd. $\square$

Remark 2.5. Without loose of generality one can assume that $b = 1$.

Lemma 2.6. *Consider the system*

$$\begin{cases} \dot{x} &= ay^{2p-1} + \hat{P}(x, y) \\ \dot{y} &= x^{2p-1} + \hat{Q}(x, y), \end{cases} \tag{2.10}$$

where $\hat{P}, \hat{Q}$ are as in the lemma 2.3. If the origin is a pure monodromic critical point of the system, then $a < 0$.

Proof. Let $P_m = \sum_{j=0}^{m} \alpha_j x^j y^{m-j}$, $Q_m = \sum_{j=0}^{m} \beta_j x^j y^{m-j}$. Suppose that $a > 0$, then Eq. (2.3) becomes

$$\begin{array}{rcl} \cos^{2p}\theta + \cos\theta\left(\sum_{j=0}^{m-1} \alpha_j \cos^j \sin^{m-j}\right) & & \\ -a\sin^{2p}\theta - \sin\theta\left(\sum_{j=1}^{m} \beta_j \cos^j \sin^{m-j}\right) & = & 0. \end{array} \tag{2.11}$$

In the interval $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ we can divide Eq. (2.11) by $\cos^{2p}(\theta)$ and we have

$$\tan^{2p}(\theta) + H(\cos\theta, \sin\theta) = 0,$$

where H is an analytic function. Let

$$G(\theta) := \tan^{2p}(\theta) + H(\cos\theta, \sin\theta).$$

In the interval $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$, $G(\theta)$ is analytic and we have $G(0) = 1$ and since $a > 0$, $\lim_{\theta \to \frac{\pi}{2}^+} G(\theta) = -\infty$ and $G(\theta)$ has at least one root in the interval, therefore the origin can not be a pure monodromic critical point. $\square$

proof of theorem 2.2: By the previous lemmas the proof is obvious.

3. Systems with conditional monodromic critical point

In this section we consider the planar polynomial systems with the origin as a conditional monodromic critical point. Using the polar blow up method we study the phase portrait of the system around the origin.

3.1. **The blow up method.** There are two basic blowing up methods, the polar blow up and the directional blow up and indeed in the plane both of them are a unique technique. In the following subsection we recall the polar blow up method briefly, for more details see [1]. The idea behind the blow up method is to explode the degenerate critical point to a line or a circle to better understanding of the local behavior around the critical point. In the following this transformation has been described.

Consider the planar system

$$\begin{cases} \dot{x} &= P(x,y) = \sum_{i=2}^{m} P_i(x,y), \\ \dot{y} &= Q(x,y) = \sum_{i=2}^{m} Q_i(x,y), \end{cases} \tag{3.1}$$

where P_i, Q_i are homogeneous polynomials of degree i. The origin is a degenerate critical point of the system. In the polar coordinates $(x, y) = (r\cos(\theta), r\sin(\theta))$ where $r \in \mathbb{R}, \theta \in [0, 2\pi)$ after canceling r^{k-1} ,where k is the least number that P_k or Q_k are not identically zero, the system (3.1) becomes

$$\begin{cases} \dot{r} &= \mathcal{R}(\theta)r + \mathcal{O}(r^2), \\ \dot{\theta} &= \mathcal{F}(\theta) + \mathcal{O}(r), \end{cases} \tag{3.2}$$

where $\mathcal{R}, \mathcal{F}$ are trigonometric polynomials.

Let θ_1, θ_2 be two consecutive roots of $\mathcal{F}$. The region limited by $r > 0$, $\theta_1 < \theta < \theta_2$ is called a sector. Let $r > 0$, $\theta_1 < \theta < \theta_2$ be a sector and $r(\theta)$ be a trajectory of the system through a point of the sector, then $r(\theta)$ is defined for every $\theta_1 < \theta < \theta_2$ and when θ approaches one of the θ_1 or θ_2, then either $r(\theta)$ approaches 0 or $+\infty$.

Definition 3.1. The sector $r > 0, \theta_1 < \theta < \theta_2$ is called

a) elliptic, if $r(\theta) \to 0$ when θ approaches θ_1^+ or θ_2^-,
b) hyperbolic, if $r(\theta) \to +\infty$ when θ approaches θ_1^+ or θ_2^-,

FIGURE 1. Possible situations for a sector after blowing up.

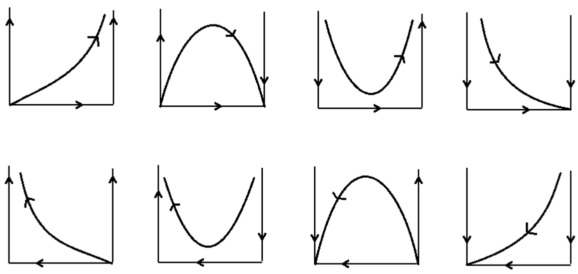
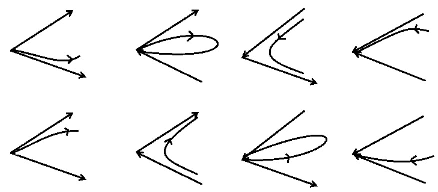


FIGURE 2. Possible situations for a sector after going back through blowing up.



c) parabolic, if $R(\theta) \to 0$ when $\theta \to \theta_1^+$ and $R(\theta) \to +\infty$ when $\theta \to \theta_2^-$ or vice versa.

Let $\theta_1 < \theta_2 < \ldots < \theta_k$ be all the roots of the equation $\mathcal{F} = 0$ in the interval $[0, 2\pi)$. For arbitrary $1 \leq i \leq k$, if $\dot{r}(\theta_i) \neq 0$ and $\dot{r}(\theta_{i+1}) \neq 0$, then the sector between θ_i, θ_{i+1} has been shown in Figure 1. In this figure we use the 'right horizontal arrow' to show that the sign of $\dot{\theta}$ in the interval (θ_i, θ_{i+1}) is positive and thus θ increase in this interval and the 'left horizontal arrow' shows that the sign of $\dot{\theta}$ in the interval (θ_i, θ_{i+1}) is negative and thus θ decrease in this interval. Similarly the vertical arrows show the sign of $\dot{r}(\theta_i)$ and $\dot{r}(\theta_{i+1})$.

Going back through the blow up, indeed by shrinking the circle $r = 0$ to a point, the sectors in Figure 1 transform to the sectors shown in Figure 2 respectively. Thus the phase portrait around the origin can be found easily and the next result is obvious.

Theorem 3.2. *Let the origin be a monodromic critical point of the system (3.1), which in the polar coordinates becomes the system (3.2). Then either $\mathcal{F}(\theta)$ has no roots in the interval $[0, 2\pi)$, or for every roots of $\mathcal{F}(\theta)$ we have $\dot{r}(\theta) = 0$.*

FIGURE 3. Sectors of the system (3.3) after blowing up.

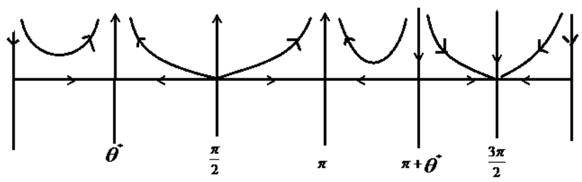
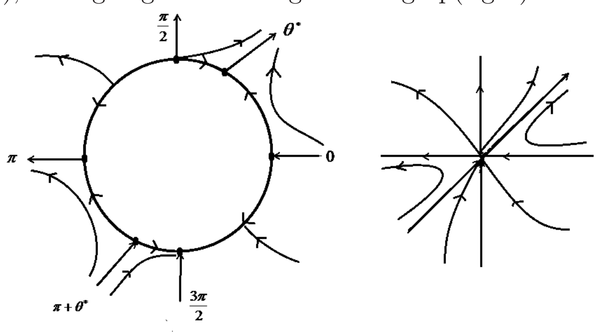


FIGURE 4. Sectors of the system (3.3) after blowing up on the cylinder (left), after going back through blowing up(right).



Example 3.3. *Consider the planar system*

$$\left\{\begin{array}{lcl} \dot{x} & = & -x^2+3xy, \\ \dot{y} & = & xy+2y^2, \end{array}\right. \tag{3.3}$$

In polar coordinates and after canceling r the system becomes

$$\left\{\begin{array}{lcl} \dot{r} & = & r\left(-\cos^3\theta+3\cos^2\theta\sin\theta+2\sin^3\theta+\cos\theta\sin^2\theta\right), \\ \dot{\theta} & = & \cos\theta\sin\theta(2\cos\theta-\sin\theta). \end{array}\right. \tag{3.4}$$

On the line $r=0$ the critical points of the system (3.4) are

$$(0,\frac{\pi}{2}),(0,\pi),(0,\frac{3\pi}{2}),(0,\theta^*),(0,\pi+\theta^*),$$

where $\tan(\theta^*)=2$. The phase portrait of the system on the cylinder is shown in Figure 3. Identifying $\theta=0$ and $\theta=2\pi$ the exceptional divisor can be considered as a circle. Shrinking the circle to a point is going back through the blow up, Figure 4.

Remark 3.4. If for every root θ^* of $\mathcal{F}$, $\dot{r}(\theta^*) = 0$, then the polar blow up method gives no information about monodromy.

Conjecture: If the origin is a monodromic critical point of the system (1.1), then there is an analytic change of variables $(x, y) = (f(r, \theta), g(r, \theta))$ such that in the coordinate system (r, θ) there is no characteristic directions.

4. Examples

Here we give two examples of the systems with conditional monodromic critical point and give some appropriate 'quasi-polar' coordinates at which the origin is a pure monodromic critical point. By the quasi-polar coordinates for the system

$$\begin{cases} \dot{x} &=& -y^{2p-1} + \mathcal{O}(y^{2p}) + xP(x,y), \\ \dot{y} &=& x^{2q-1} + \mathcal{O}(x^{2q}) + yQ(x,y), \end{cases} \tag{4.1}$$

we mean the change of variables

$$x = r^p Cs\theta, \quad y = r^q Sn\theta, \tag{4.2}$$

where Cs, Sn are the solution of the Hamiltonian system

$$\begin{cases} \dot{x} &=& -y^{2p-1}, \\ \dot{y} &=& x^{2q-1}. \end{cases} \tag{4.3}$$

So we have

$$\dot{r} = \frac{x^{2q-1}\dot{x} + y^{2p-1}\dot{y}}{pqr^{2pq-1}}, \quad \dot{\theta} = \frac{px\dot{y} - qy\dot{x}}{pqr^{p+q}}.$$

For more details on quasi-polar coordinates see for instance [2].

Consider the system

$$\begin{cases} \dot{x} &=& y \\ \dot{y} &=& f(x) + \phi(x)y, \end{cases} \tag{4.4}$$

where $f(x) = ax^\alpha + \mathcal{O}(x^{\alpha+1})$ and $\phi(x) = bx^\beta + \mathcal{O}(x^{\beta+1})$.

As a consequence of Andreev Theorem, the origin is a monodromic critical point of (4.4) if a is negative, $\alpha = 2n - 1$ and one of the following three conditions holds

(1) $\beta > n - 1$,
(2) $\beta = n - 1$ and $b^2 + 4an < 0$,
(3) $\phi \equiv 0$, [3].

Example 4.1. *By the above descriptions the origin is a monodromic nilpotent critical point for the system*

$$\begin{cases} \dot{x} &= y, \\ \dot{y} &= -x^3 + x^5 y. \end{cases} \tag{4.5}$$

In the polar coordinates the system (4.5) becomes

$$\begin{cases} \dot{r} &= r\cos\theta\sin\theta + r^3\cos^3\theta\sin\theta + r^6\cos^5\theta\sin^2\theta, \\ \dot{\theta} &= \sin^2\theta - r^2\cos^4\theta + r^5\cos^6\theta\sin\theta. \end{cases} \tag{4.6}$$

In the system (4.6) the equation $\mathcal{F} = 0$ has two roots in the interval $[0, 2\pi)$. These roots are $0, \pi$, and $\dot{r}(0) = \dot{r}(\pi) = 0$ and we can't find anything about the local behavior of the system. Using Lyapunov polar coordinate system $(x, y) = (r\ Cs\theta, r^2\ Sn\theta)$ and cancel r, the system (4.5) becomes

$$\begin{cases} \dot{r} &= r^5\ Cs^5\theta\ Sn^2\theta, \\ \dot{\theta} &= -1 + r^4\ Cs^6\theta\ Sn\theta, \end{cases} \tag{4.7}$$

where $Cs\theta, Sn\theta$ are the solution of the system

$$\begin{cases} \dot{x} &= -y, \\ \dot{y} &= x^3. \end{cases} \tag{4.8}$$

The system (4.7) has no characteristic directions. Thus the origin is a monodromic critical point of the system (4.5).

Example 4.2. *Consider the system*

$$\begin{cases} \dot{x} &= -y^3 + xy^2 + x^2 y, \\ \dot{y} &= x^5 + 2xy^2 + y^3, \end{cases} \tag{4.9}$$

where in the polar coordinates after canceling r^2 becomes

$$\begin{cases} \dot{r} &= r(\cos^2\theta\sin^2\theta + \cos^3\theta\sin\theta + \cos\theta\sin^3\theta + \sin^4\theta) + r^3\cos^5\theta\sin\theta, \\ \dot{\theta} &= \sin^2\theta + r^2\cos^6\theta. \end{cases} \tag{4.10}$$

We have $\mathcal{F}(\theta) = \sin^2\theta$ and this system has two characteristic directions in the interval $[0, 2\pi)$ i.e. $0, \pi$, and $\dot{r}(0) = \dot{r}(\pi) = 0$.

Now we use the quasi polar coordinates $(x, y) = (r^2 Cs\theta, r^3 Sn\theta)$ at which Cs, Sn are the solutions of the system

$$\begin{cases} \dot{x} &= -y^3, \\ \dot{y} &= x^5. \end{cases}$$

One can show easily that

$$\dot{r} = \frac{x^5\dot{x} + y^3\dot{y}}{6r^{11}}, \quad \dot{\theta} = \frac{2x\dot{y} - 3y\dot{x}}{6r^5}.$$

In this coordinates after canceling r^7 the system (4.9) becomes

$$\begin{cases} \dot{r} &= \frac{1}{6r}(2Cs\theta\ Sn^5\theta + Cs^7\theta\ Sn\theta) + \frac{1}{6}(Cs^6\theta\ Sn^2\theta + Sn^6\theta), \\ \dot{\theta} &= 1 - \frac{1}{6r}Cs\theta\ Sn^3\theta + \frac{1}{6r^2}Cs^2\theta\ Sn^2\theta. \end{cases} \tag{4.11}$$

The system (4.11) has no characteristic directions, thus the origin is monodromic.

Example 4.3. *Consider the planar system*

$$\begin{cases} \dot{x} &= -y^3 + \sum_{i=4}^{n} P_i(x,y), \\ \dot{y} &= x^3 + \sum_{i=4}^{n} Q_i(x,y), \end{cases} \tag{4.12}$$

where P_i, Q_i are homogeneous polynomials of degree i and $n \in \mathbb{N}$. This system has no characteristic direction, because in the polar coordinates we have

$$\mathcal{F}(\theta) = cos^4\theta + \sin^4\theta > 0.$$

Thus the origin is a monodromic critical point.

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