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Numerical inversion of Laplace transform via wavelet in ordinary differential equations

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Abstract This paper presents a rational Haar wavelet operational method for solving the inverse Laplace transform problem and improves inherent errors from irrational Haar wavelet. The approach is thus straightforward, rather simple and suitable for computer programming. We define that P is the operational matrix for integration of the orthogonal Haar wavelet. Simultaneously, simplify the formulaes of listing table to a minimum expression and obtain the optimal operation speed. The local property of Haar wavelet is fully applied to shorten the calculation process in the task.

Keywords. Haar wavelet, Inverse Laplace transform, Operational matrix of integration, Haar product matrix.

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1. Introduction

To solve differential equation and corresponding initial and boundary value problem, Laplace transformation reduces the problem of differential equation to an algebraic problem. The advantage of Laplace transformation is that it solves initial value problem without first determining a general solution and it solves nonhomogeneous equations without first solving the corresponding homogeneous equation [11].

When we evaluate many engineering systems described by differential equations via Laplace transformation, fractional functions or transcendental functions of s will be formed. For example, problems in thermal processes in hole diffusion of transistors, in electromagnetic devices, in transmission lines and in percolation processes often have mathematical models involving $\sqrt{s}$, $\sqrt{s^2+1}$, $e^{-\sqrt{s}}$ etc. Developments in this field can be founded in Pade's approximation [13]; Carlson and Halijak's approach [2] using regular Newton's process; Lerner's work on partial analogue approximation

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[12]; kiloneitreva and Netushil's technique [10] developing special functions; Chen and Chiu's use the Fast Fourier transform [3].

In this paper, we present a rational Haar wavelet operational method for solving the inverse Laplace transform problem and improves inherent errors from irrational Haar wavelet. Simultaneously, simplify the formulaes of listing table to a minimum expression and obtain the optimal operation speed. The local property of Haar wavelet is fully applied to shorten the calculation process in the task.

2. Some properties of Haar wavelets

The orthogonal set of Haar wavelets $h_i(t)$ is a group of square waves with magnitude of ± 1 in certain intervals and zeros elsewhere. The first curve is $h_0(t)$. The second curve $h_1(t)$ is the fundamental square wave.

$$h_0(t) = 1,\ 0 \le t < 1, \quad h_1(t) = \left\{ \begin{array}{cc} 1, & 0 \le t < \frac{1}{2} \\ -1, & \frac{1}{2} \le t < 1 \end{array} \right\}, \tag{2.1}$$

$$h_n(t) = h_1\left(2^j t - k\right),\ n = 2^j + k,\ j \ge 0,\ 0 \le k < 2^j. \tag{2.2}$$

Any square integrable function $y(t)$ in the interval $[0,1)$ can be expanded by a Haar series of infinite terms

$$y(t) = \sum_{i=0}^{\infty} c_i\, h_i(t),\ i = 2^j + k,\ j \ge 0,\ 0 \le k < 2^j, \tag{2.3}$$

where the Haar coefficients

$$c_i = 2^j \int_0^1 y(t)\ h_i(t)\, dt, \tag{2.4}$$

are determined such that the following integral square error ϵ is minimized:

$$\epsilon = \int_0^1 [y(t) - \sum_{i=0}^{m-1} c_i\, h_i(t)]^2 dt,\ m = 2^j,\ j \in \{0\} \cup \mathbf{N}. \tag{2.5}$$

Usually, the series expansion of (2.4) contains infinite terms for smooth $y(t)$. If $y(t)$ is piecewise constant by itself, or may be approximated as piecewise constant during each subinterval, then (2.4) will be terminated at finite m terms, that is

$$y(t) \approx \sum_{i=0}^{m-1} c_i\, h_i(t) = \mathbf{c}_{(m)}^T \mathbf{h}_{(m)}(t), \tag{2.6}$$

$$\mathbf{c}_{(m)} \triangleq [c_0\ c_1\ \cdots\ c_{m-1}]^T, \tag{2.7}$$

$$\mathbf{h}_{(m)}(t) \triangleq [h_0(t)\ \ h_1(t)\ \ \cdots\ \ h_{m-1}(t)]^T, \tag{2.8}$$

where "T" means transpose, $m = 2^j$, $j \in \{0\} \cup \mathbf{N}$, and the subscript m in the parentheses denotes their dimensions.

2.1. **Local basis.** As defined in (2.3), each Haar wavelet contains just one wavelet during some interval, and remains to be zero elsewhere in the interval $[0, 1)$. Therefore, Haar set forms a local basis. These zeros make Haar transform (HT) much easier and faster than others, such as fast Fourier transform (FFT) and Walsh transform (WT). The numbers of additions and multiplications for these three transforms are shown in Table I. The fast capability of HT should be impressive.

2.2. **Piecewise constant approximation.** As mentioned in section one, a piecewise constant function can be expanded into Haar series with finite terms. Same situations exist for Walsh transform and block pulse transform (BPT). Therefore, each Walsh function can be expanded into Haar series with m terms exactly with zero truncation error, and *vice versa* (from now on, subscript m is omitted when no confusions are likely to arise).

$$\mathbf{w}(t) = T_{WH} \cdot \mathbf{h}(t), \ \ \mathbf{b}(t) = T_{BH} \cdot \mathbf{h}(t), \tag{2.9}$$

$$\mathbf{h}(t) = T_{HW} \cdot \mathbf{w}(t), \ \ \mathbf{h}(t) = T_{HB} \cdot \mathbf{b}(t), \tag{2.10}$$

where $\mathbf{h}(t)$, $\mathbf{w}(t)$, and $\mathbf{b}(t)$ are Haar, Walsh, and block pulse vectors respectively. If any smooth function $y(t)$ is expanded into Haar series with m terms, the truncation error ϵ_H is given by (2.6).

$$y(t) \approx \mathbf{c_h}^T \cdot \mathbf{h}(t), \tag{2.11}$$

$$\epsilon_H = \int_0^1 [y(t) - \mathbf{c_h}^T \cdot \mathbf{h}(t)]^2 dt. \tag{2.12}$$

If $y(t)$ is expanded into Walsh series, the truncation error is ϵ_W.

$$y(t) \approx \mathbf{c_w}^T \cdot \mathbf{w}(t) = \mathbf{c_w}^T \cdot T_{WH} \cdot \mathbf{h}(t), \tag{2.13}$$

$$\epsilon_W = \int_0^1 [y(t) - \mathbf{c_w}^T \cdot \mathbf{w}(t)]^2 dt = \int_0^1 [y(t) - \mathbf{c_w}^T \cdot T_{WH} \cdot \mathbf{h}(t)]^2 dt. \tag{2.14}$$

The series expansion must be unique. Equating the right-hand sides of (2.12) and (2.14), we have

$$\mathbf{c_w}^T \cdot T_{WH} = \mathbf{c_h}^T, \tag{2.15}$$

$$\epsilon_W = \epsilon_H. \tag{2.16}$$

Similarly, we can prove $\epsilon_H = \epsilon_B$. Therefore HT, WT, and BPT have the same truncation errors, the same accuracies, the same scale resolutions, if the same $m = 2^j$ is used, since they are all piecewise approximation.

2.3. **Integration of Haar wavelets.** In the wavelet analysis for a dynamic system, all functions need to be transformed into Haar series. Since the differentiation of Haar wavelets always results in impulse functions which should be avoided, the integration of Haar wavelets is preferred, which should be expandable into Haar series with Haar coefficient matrix P [4,5,8,9].

$$\int_0^t \mathbf{h}_{(m)}(\tau)\, d\tau \approx P_{(m\times m)}\mathbf{h}_{(m)}(t), \ t \in [0,1), \tag{2.17}$$

where the m-square matrix P is called the *operational matrix* of integration which satisfies the following recursive formula.

$$P_{(m\times m)} = \frac{1}{2m}\begin{bmatrix} 2mP_{\left(\frac{m}{2}\times\frac{m}{2}\right)} & -H_{\left(\frac{m}{2}\times\frac{m}{2}\right)} \\ H^{-1}_{\left(\frac{m}{2}\times\frac{m}{2}\right)} & 0_{\left(\frac{m}{2}\times\frac{m}{2}\right)} \end{bmatrix}, \ P_{(1\times 1)} = \frac{1}{2}, \tag{2.18}$$

where $H_{(m\times m)} \triangleq [\mathbf{h}_{(m)}(t_0)\ \ \mathbf{h}_{(m)}(t_1)\ \cdots\ \mathbf{h}_{(m)}(t_{m-1})]$, $\frac{i}{m} \le t_i < \frac{i+1}{m}$, and $H^{-1}_{(m\times m)} = \frac{1}{m}H^T_{(m\times m)}\mathrm{diag}(\mathbf{r})$, $\mathbf{r} \triangleq [1\ 1\ 2\ 2\ 4\ 4\ 4\ 4\ \cdots\ \underbrace{\frac{m}{2}\ \frac{m}{2}\ \cdots\ \frac{m}{2}}_{\frac{m}{2}\ \text{elements}}]^T$, for $m > 2$.

3.4. **Multiplication of Haar wavelets.** Two basic multiplication properties of Haar wavelets are as follows:

(i) For any two Haar wavelets $h_n(t)$ and $h_l(t)$ with $n < l$.

$$h_n(t)\, h_l(t) = \rho h_l(t), \tag{2.19}$$

$$\rho = h_n\left(2^{-i}\left(q + \frac{1}{2}\right)\right) = \begin{cases} 1, & 2^{i-j}k \le q < 2^{i-j}\left(k+\frac{1}{2}\right), \\ -1, & 2^{i-j}\left(k+\frac{1}{2}\right) \le q < 2^{i-j}(k+1), \\ 0, & \text{otherwise,} \end{cases} \tag{2.20}$$

where

$$\begin{cases} n = 2^j + k, & j \geq 0, \quad 0 \leq k < 2^j. \\ l = 2^i + q, & i \geq 0, \quad 0 \leq q < 2^i. \end{cases} \tag{2.21}$$

(ii) The square of any Haar wavelet is a block pulse with magnitude of 1 during both positive and negative half waves.

In the study of linear time-varying system via Haar wavelets, it usually needs to evaluate $\mathbf{h}_{(m)}(t)\,\mathbf{h}_{(m)}^T(t)$. Let us define

$$\mathbf{h}_{(m)}(t)\,\mathbf{h}_{(m)}^T(t) \triangleq M_{(m\times m)}(t), \tag{2.22}$$

$$\begin{aligned} \mathbf{h}_a(t) &\triangleq [h_0(t)\ \ h_1(t)\ \ \cdots\ \ h_{\frac{m}{2}-1}(t)]^T = \mathbf{h}_{(\frac{m}{2})}(t), \\ \mathbf{h}_b(t) &\triangleq [h_{\frac{m}{2}}(t)\ \ h_{\frac{m}{2}+1}(t)\ \ \cdots\ \ h_{m-1}(t)]^T. \end{aligned} \tag{2.23}$$

$M(t)$ is the *Haar product matrix*, which satisfies the following recursive formula (2.25) and the coefficient relation (2.26)

$$\begin{aligned} M_{(m\times m)}(t) &= \begin{bmatrix} M_{(\frac{m}{2}\times\frac{m}{2})}(t) & H_{(\frac{m}{2}\times\frac{m}{2})}\mathrm{diag}[\mathbf{h}_b(t)] \\ \mathrm{diag}[\mathbf{h}_b(t)]H^T_{(\frac{m}{2}\times\frac{m}{2})} & \mathrm{diag}[H^{-1}_{(\frac{m}{2}\times\frac{m}{2})}\mathbf{h}_a(t)] \end{bmatrix}, \\ M_{(1\times 1)}(t) &= h_0(t), \end{aligned} \tag{2.24}$$

$$M_{(m\times m)}(t)\,\mathbf{c}_{(m)} = C_{(m\times m)}\mathbf{h}_{(m)}(t), \tag{2.25}$$

where

$$\begin{aligned} C_{(m\times m)} &= \begin{bmatrix} C_{(\frac{m}{2}\times\frac{m}{2})} & H_{(\frac{m}{2}\times\frac{m}{2})}\mathrm{diag}[\mathbf{c}_b] \\ \mathrm{diag}[\mathbf{c}_b]H^{-1}_{(\frac{m}{2}\times\frac{m}{2})} & \mathrm{diag}[\mathbf{c}_a^T H_{(\frac{m}{2}\times\frac{m}{2})}] \end{bmatrix}, \\ C_{(1\times 1)} &= c_0, \end{aligned} \tag{2.26}$$

$$\mathbf{c}_a \triangleq [c_0\ c_1\ \ \cdots\ \ c_{\frac{m}{2}-1}]^T = \mathbf{c}_{(\frac{m}{2})}, \quad \mathbf{c}_b \triangleq [c_{\frac{m}{2}}\ \ c_{\frac{m}{2}+1}\ \ \cdots\ \ c_{m-1}]^T. \tag{2.27}$$

Equation (2.26) is a very important for solving linear time-varying, bilinear, and nonlinear problems.

3. Numerical inversion of Laplace transform via rational Haar wavelet operational method

The aim of this section is to establish the procedure of inverse Laplace transform via the operational matrix P of integration of rational Haar wavelets for solving the differential equation. First, we consider a time-invariant differential equation with the initial condition as follows.

$$\dot{y}\,(t) + a\;y\,(t) = 0,\;\; y\,(0) = b. \tag{3.1}$$

Let $Y\,(s) = L[y\,(t)]$ be the Laplace transform of the unknown solution $y\,(t)$.

$$L[\dot{y}\,(t)] = sY\,(s) - y\,(0) = sY\,(s) - b. \tag{3.2}$$

Substituting (3.2) into the Laplace transform of (3.1) yields

$$sY\,(s) + aY\,(s) = b, \tag{3.3}$$

and we get the transfer function

$$Y\,(s) = \frac{b}{s+a}. \tag{3.4}$$

Equation (3.4) can be rewritten as

$$Y\,(s) = \frac{\frac{b}{s}}{1+\frac{a}{s}} \triangleq \hat{Y}\left(\frac{1}{s}\right). \tag{3.5}$$

The integration in the time domain is corresponding to multiplication of $\frac{1}{s}$ in the s domain. From the definition of the operational matrix P of integration, the integration in the time domain is equivalent to replacing $\frac{1}{s}$ by the operation matrix P in the equivalent matrix. Integration of (3.1) yields

$$y\,(t) + a\int_0^t y\,(\tau)\,d\tau = b. \tag{3.6}$$

The discrete form of (3.6) is given by

$$y^T\,(t) + a\int_0^t y^T\,(\tau)\,d\tau = bi^T\,(t)\,, \tag{3.7}$$

where $y\,(t) = [y_0\;\, y_1\;\; \cdots\;\; y_{m-1}]^T$ and $i\,(t) = [\;\underbrace{1\;1\;\cdots\;1}_{m-1\;\text{ elements}}\;]^T$ are column vectors. Assuming $y^T = c^T H$ and replacing the integral sign by the matrix of integration P in (3.7), we obtain $c^T[I + aP]H = b[\;\underbrace{1\;1\;\cdots\;1}_{m-1\;\text{ elements}}\;]$ where I is the identity matrix with the dimension $m \times m$. Therefore, c^T is given by

$$\begin{aligned} c^T &= [\ \underbrace{1\,1\ \cdots\ 1}_{m-1 \text{ elements}}\ ]H^{-1}[bI][I+aP]^{-1} \\ &= [\ \underbrace{1\,1\ \cdots\ 1}_{m-1 \text{ elements}}\ ]H^{-1}P^{-1}[bP][I+aP]^{-1} \\ &= [\ \underbrace{1\,1\ \cdots\ 1}_{m-1 \text{ elements}}\ ]H^{-1}P^{-1}\hat{Y}\,(P)\,. \end{aligned} \tag{3.8}$$

However, from $P = HP_BH^{-1}$, we obtain $P^{-1} = HP_B^{-1}H^{-1}$. Substituting it into (3.8) yields

$$\begin{aligned} c^T &= [\ \underbrace{1\,1\ \cdots\ 1}_{m-1 \text{ elements}}\ ]P_B^{-1}H^{-1}\hat{Y}\,(P) \\ &= [\ \underbrace{0\,0\ \cdots\ 0}_{\frac{m}{2} \text{ elements}}\ \ \underbrace{2m\ 2m\ \cdots\ 2m}_{\frac{m}{2} \text{ elements}}]\hat{Y}\,(P)\,. \end{aligned} \tag{3.9}$$

y^T, the inversion of the Laplace transform $Y\,(s)$, is given by

$$y^T = c^T H = [\ \underbrace{0\,0\ \cdots\ 0}_{\frac{m}{2} \text{ elements}}\ \ \underbrace{2m\ 2m\ \cdots\ 2m}_{\frac{m}{2} \text{ elements}}]\hat{Y}\,(P)\,H \triangleq k^T\hat{Y}\,(P)\,H. \tag{3.10}$$

The solution given by (3.10) is superior to a proposition of [6] and much simpler compared with those from previous literature [1,7]. Numerical inversions of Laplace transform in terms of operational matrix are shown in Table II.

4. Numerical results

We apply the method presented in this paper to solve the following four examples. Our method will be used as a basis for comparison because it differs from the elimination and substitution algorithm used in [4,5] and the single-term method given in [8].

Example 1. Consider the following linear ordinary differential equation of order 1:

$$y'(t) + y(t) = t \tag{4.1}$$

with initial condition $y(0) = 1$, and the exact solution $y(t) = 2\exp(-t) - 1 + t$. The comparison among our method, single-term method and the exact solution for $m = 64$ and $t \in [0, 2)$ is shown in Table III and Figure 1, which confirms that the new approach gives almost the same solution as the exact solution with respect to numerical solutions. The average relative errors of our method and the single-term method are $5.784539985255178 \cdot 10^{-5}$ and $5.784539985097847 \cdot 10^{-5}$, respectively.

Example 2. Consider the following linear ordinary differential equation of order 2:

$$y''(t) + 2y'(t) + y(t) = 1 \tag{4.2}$$

with initial conditions $y(0) = 3$ and $y'(0) = 4$, and the exact solution $y(t) = 1 + 2\exp(-t) + 6t\exp(-t)$, the result for $m = 256$ and $t \in [0, 8)$ is shown in Table IV and Figure 2. The average relative errors of our method and the single-term method are $3.059437400552340 \cdot 10^{-5}$ and $3.059437400711908 \cdot 10^{-5}$, respectively.

Example 3. Consider the following linear ordinary differential equation of order 3:

$$y'''(t) + 3y''(t) + 3y'(t) + y(t) = t^3 \exp(-t) \tag{4.3}$$

with initial conditions $y(0) = 3$, $y'(0) = 1$ and $y''(0) = -2$, and the exact solution $y(t) = (3 + 4t + 3t^2/2 + t^6/5!)\exp(-t)$, the result for $m = 256$ and $t \in [0, 8)$ is shown in Table V and Figure 3. The average relative errors of our method and the single-term method are $2.835591714181639 \cdot 10^{-5}$ and $2.731714434050271 \cdot 10^{-5}$, respectively.

Example 4. Consider the following linear ordinary differential equation of order 4:

$$y''''(t) + 2\,a^2\,y''(t) + a^4\,y(t) = \cos(at),\ a = 10^{-3} \tag{4.4}$$

with initial conditions $y(0) = 0$, $y'(0) = 0$, $y''(0) = 0$ and $y'''(0) = 0$, and the exact solution $y(t) = \big(t\sin(at) - a\,t^2\cos(at)\big)/(8\,a^3)$, the result for $m = 512$ and $t \in [0, 16)$ is shown in Table VI and Figure 4. The average relative errors of our method and the single-term method are 0.00640536141138 and 0.00640539033849, respectively.

The differences between these two approaches are indiscernible, since they have nearly the same accuracy. The final state variables and partial computation time of these methods have been put together in Tables 3–6 for comparison. As shown above, our method is superior to the single-term method in accuracy for some numerical solutions. This impressive achievement should be attributed to the local orthogonal properties of Haar wavelets.

5. Conclusions

The main contributions should be ascribed to the nice local orthogonal Haar wavelets. In this paper, a rational Haar wavelet operational method for solving the inverse Laplace transform problem and improves inherent errors from irrational Haar wavelet is presented. The approach is thus straightforward, rather simple and suitable for computer programming. A new method for calculating the inverse Laplace transform is derived, based on the derived operational matrix of the Haar wavelets. Simultaneously, simplify the formulaes of listing table to a minimun expression and obtain the optimal operation speed. The local property of Haar wavelet is fully applied to shorten the calculation process in the task.

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