



A new fractional sub-equation method for solving the space-time fractional differential equations in mathematical physics

Mehmet Ekici*

Department of Mathematics,
Faculty of Science and Arts,
Bozok University, 66100 Yozgat, Turkey
E-mail: mehmet.ekici@bozok.edu.tr

Elsayed M. E. Zayed

Department of Mathematics,
Faculty of Science, Zagazig University,
P.O. Box 44519, Zagazig, Egypt
E-mail: e.m.e.zayed@hotmail.com

Abdullah Sonmezoglu

Department of Mathematics,
Faculty of Science and Arts,
Bozok University, 66100 Yozgat, Turkey
E-mail: abduallah.sonmezoglu@bozok.edu.tr

Abstract

In this paper, a new fractional sub-equation method is proposed for finding exact solutions of fractional partial differential equations (FPDEs) in the sense of modified Riemann-Liouville derivative. With the aid of symbolic computation, we choose the space-time fractional Zakharov-Kuznetsov-Benjamin-Bona-Mahony (ZKBBM) equation in mathematical physics with a source to illustrate the validity and advantages of the novel method. As a result, some new exact solutions including solitary wave solutions and periodic wave solutions are successfully obtained. The proposed approach can also be applied to other nonlinear FPDEs arising in mathematical physics.

Keywords. Fractional sub-equation method, fractional partial differential equations, exact solutions, modified Riemann-Liouville derivative.

2010 Mathematics Subject Classification. 35K99, 35P05, 35P99.

1. INTRODUCTION

Nonlinear phenomena appear in a wide variety of scientific applications such as plasma physics, solid state physics and fluid dynamics. Fractional partial differential equations (FPDEs) have been attracted great interest due to their various applications in the areas of physics, biology, engineering, signal processing, control theory, finance and fractal dynamics [19, 26, 27].

Received: 7 December 2014; Accepted: 17 February 2015.

* Corresponding author.

Recently, several powerful methods have been proposed to obtain approximate and exact solutions of FPDEs, such as the Adomian decomposition method [5, 29], the variational iteration method [6, 14, 33], the homotopy analysis method [1, 3, 28, 30], the homotopy perturbation method [7, 8, 10], the Lagrange characteristic method [17], the fractional sub-equation method [42], the (G'/G) -expansion method [43, 44], the first integral method [20], the transformed rational function method [23], the multiple exp-function method [24, 25], the generalized Riccati equation method [21], the Frobenius decomposition technique [22], the local fractional differential equations [34, 35], the local fractional variation iteration method [36], local fractional Fourier series method [13], the Cantor-type cylindrical coordinate method [37], the Yang-Fourier and Yang-Laplace transforms [12], the fractional complex transform method [31], the modified simple equation method [15, 39, 40, 41].

In [16], Jumarie proposed a modified Riemann-Liouville derivative. With this kind of fractional derivatives and some useful formulas, we can convert FPDEs into ordinary differential equations (ODEs) with integer orders by applying suitable transformations.

In this paper, we propose a new fractional sub-equation method to establish exact solutions for FPDEs in the sense of modified Riemann-Liouville derivative defined by Jumarie [16]. This method is a fractional version of the known extended (G'/G) -expansion method [4, 9, 11, 38]. The proposed approach is based on the following fractional ODE:

$$D_{\xi}^{2\alpha}G(\xi) + \mu G(\xi) = 0, \quad (1.1)$$

where μ is a constant and $D_{\xi}^{\alpha}G(\xi)$ denotes the modified Riemann-Liouville derivative of order α for $G(\xi)$ with respect to ξ .

The paper is arranged as follows: In Section 2, we give some definitions and properties of Jumarie's modified Riemann-Liouville derivative. We also give the expression for $\frac{D_{\xi}^{\alpha}G(\xi)}{G(\xi)}$ related to Eq. (1.1). In Section 3, we present the main steps of the fractional sub-equation method for solving FPDEs. In Section 4, we apply this method to construct exact solutions of the space-time fractional ZKBBM equation. We include figures to show the properties of some solutions of this equation. Finally, we summarize our results in the conclusion section.



2. JUMARIE'S MODIFIED RIEMANN-LIOUVILLE DERIVATIVE AND GENERAL EXPRESSION FOR $\frac{D_t^\alpha G(\xi)}{G(\xi)}$

Jumarie's modified Riemann-Liouville derivative of order α is defined by the following expression [16]:

$$D_t^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-\xi)^{-\alpha} (f(\xi) - f(0)) d\xi, & 0 < \alpha < 1, \\ (f^{(n)}(t))^{(\alpha-n)}, & n \leq \alpha < n+1, n \geq 1. \end{cases} \tag{2.1}$$

We list some important properties for the modified Riemann-Liouville derivative as follows [16]:

$$D_t^\alpha t^r = \frac{\Gamma(1+r)}{\Gamma(1+r-\alpha)} t^{r-\alpha}, \tag{2.2}$$

$$D_t^\alpha (f(t)g(t)) = g(t)D_t^\alpha f(t) + f(t)D_t^\alpha g(t), \tag{2.3}$$

$$D_t^\alpha f [g(t)] = f'_g [g(t)] D_t^\alpha g(t) = D_g^\alpha f [g(t)] (g'(t))^\alpha. \tag{2.4}$$

In order to obtain the general solutions for Eq. (1.1), we suppose $G(\xi) = H(\eta)$ and a nonlinear fractional complex transformation $\eta = \frac{\xi^\alpha}{\Gamma(1+\alpha)}$. Then by Eq. (2.2), the first equality in Eq. (2.4) and definition of Principle of Derivative Increasing Orders [18], Eq. (1.1) can be turned into the following second ordinary differential equation

$$H''(\eta) + \mu H(\eta) = 0. \tag{2.5}$$

By the general solutions of Eq. (2.5), we have

$$\frac{H'(\eta)}{H(\eta)} = \begin{cases} \frac{\sqrt{-\mu} \frac{A_1 \sinh(\sqrt{-\mu}\eta) + A_2 \cosh(\sqrt{-\mu}\eta)}{A_1 \cosh(\sqrt{-\mu}\eta) + A_2 \sinh(\sqrt{-\mu}\eta)}, & \mu < 0, \\ \sqrt{\mu} \frac{A_1 \cos(\sqrt{\mu}\eta) - A_2 \sin(\sqrt{\mu}\eta)}{A_1 \sin(\sqrt{\mu}\eta) + A_2 \cos(\sqrt{\mu}\eta)}, & \mu > 0, \\ \frac{A_1}{A_1\eta + A_2}, & \mu = 0, \end{cases} \tag{2.6}$$



where A_1, A_2 are arbitrary constants.

Since $D_\xi^\alpha G(\xi) = D_\xi^\alpha H(\eta) = H'(\eta)D_\xi^\alpha \eta = H'(\eta)$, we obtain

$$\frac{D_\xi^\alpha G(\xi)}{G(\xi)} = \begin{cases} \sqrt{-\mu} \frac{A_1 \sinh\left(\frac{\sqrt{-\mu}}{\Gamma(1+\alpha)}\xi^\alpha\right) + A_2 \cosh\left(\frac{\sqrt{-\mu}}{\Gamma(1+\alpha)}\xi^\alpha\right)}{A_1 \cosh\left(\frac{\sqrt{-\mu}}{\Gamma(1+\alpha)}\xi^\alpha\right) + A_2 \sinh\left(\frac{\sqrt{-\mu}}{\Gamma(1+\alpha)}\xi^\alpha\right)}, & \mu < 0, \\ \sqrt{\mu} \frac{A_1 \cos\left(\frac{\sqrt{\mu}}{\Gamma(1+\alpha)}\xi^\alpha\right) - A_2 \sin\left(\frac{\sqrt{\mu}}{\Gamma(1+\alpha)}\xi^\alpha\right)}{A_1 \sin\left(\frac{\sqrt{\mu}}{\Gamma(1+\alpha)}\xi^\alpha\right) + A_2 \cos\left(\frac{\sqrt{\mu}}{\Gamma(1+\alpha)}\xi^\alpha\right)}, & \mu > 0, \\ \frac{A_1 \Gamma(1+\alpha)}{A_1 \xi^\alpha + A_2 \Gamma(1+\alpha)}, & \mu = 0. \end{cases} \quad (2.7)$$

3. DESCRIPTION OF THE FRACTIONAL SUB-EQUATION METHOD

In this section, we describe the main steps of the fractional sub-equation method for finding exact solutions of FPDEs.

Suppose that a fractional partial differential equation, say in the independent variables t, x_1, x_2, \dots, x_n , is given by

$$P(u_1, \dots, u_k, D_t^\alpha u_1, \dots, D_t^\alpha u_k, D_{x_1}^\alpha u_1, \dots, D_{x_1}^\alpha u_k, \dots, D_{x_n}^\alpha u_1, \dots, D_{x_n}^\alpha u_k, D_t^{2\alpha} u_1, \dots, D_t^{2\alpha} u_k, D_{x_1}^{2\alpha} u_1, \dots) = 0, \quad (3.1)$$

where $u_i = u_i(t, x_1, x_2, \dots, x_n), i = 1, \dots, k$ are unknown functions, P is a polynomial in u_i and their various partial derivatives including fractional derivatives.

Step 1. Suppose that

$$u_i(t, x_1, x_2, \dots, x_n) = U_i(\xi), \quad \xi = ct + k_1 x_1 + k_2 x_2 + \dots + k_n x_n + \xi_0. \quad (3.2)$$

Then by the second equality in Eq. (2.4), Eq. (3.1) can be turned into the following fractional ordinary differential equation with respect to the variable ξ :

$$\tilde{P}\left(U_1, \dots, U_k, c^\alpha D_\xi^\alpha U_1, \dots, c^\alpha D_\xi^\alpha U_k, k_1^\alpha D_\xi^\alpha U_1, \dots, k_1^\alpha D_\xi^\alpha U_k, \dots, k_n^\alpha D_\xi^\alpha U_1, \dots, k_n^\alpha D_\xi^\alpha U_k, c^{2\alpha} D_\xi^{2\alpha} U_1, \dots, c^{2\alpha} D_\xi^{2\alpha} U_k, k_1^{2\alpha} D_\xi^{2\alpha} U_1, \dots\right) = 0. \quad (3.3)$$



Step 2. Suppose that the solution of (3.3) can be expressed by a polynomial in $(D_\xi^\alpha G/G)$ as follows:

$$\begin{aligned}
 U_j(\xi) = & a_{j,0} + \sum_{i=1}^{m_j} \left[a_{j,i} \left(\frac{D_\xi^\alpha G}{G} \right)^i \right. \\
 & + b_{j,i} \left(\frac{D_\xi^\alpha G}{G} \right)^{i-1} \sqrt{\sigma \left\{ 1 + \frac{1}{\mu} \left(\frac{D_\xi^\alpha G}{G} \right)^2 \right\}} \\
 & \left. + c_{j,i} \left(\frac{D_\xi^\alpha G}{G} \right)^{-i} + d_{j,i} \frac{\left(\frac{D_\xi^\alpha G}{G} \right)^{-i+1}}{\sqrt{\sigma \left\{ 1 + \frac{1}{\mu} \left(\frac{D_\xi^\alpha G}{G} \right)^2 \right\}}} \right], \\
 & j = 1, 2, \dots, k, \tag{3.4}
 \end{aligned}$$

where $G = G(\xi)$ satisfies Eq. (1.1), σ is a constant, and $a_{j,0}, a_{j,i}, b_{j,i}, c_{j,i}, d_{j,i}, i = 1, 2, \dots, m, j = 1, 2, \dots, k$, are constants to be determined later. The positive integer m can be determined by considering the homogeneous balance between the highest-order derivatives and nonlinear terms appearing in (3.3).

Step 3. Substituting (3.4) along with Eq. (1.1) into Eq. (3.3) and collecting all the terms with the same order of $(\frac{D_\xi^\alpha G}{G})$ and $(\frac{D_\xi^\alpha G}{G}) \sqrt{\sigma \left\{ 1 + \frac{1}{\mu} \left(\frac{D_\xi^\alpha G}{G} \right)^2 \right\}}$, the left-hand side of (3.3) is converted into a polynomial in $(\frac{D_\xi^\alpha G}{G})$ and $(\frac{D_\xi^\alpha G}{G}) \sqrt{\sigma \left\{ 1 + \frac{1}{\mu} \left(\frac{D_\xi^\alpha G}{G} \right)^2 \right\}}$. Equating each coefficient of this polynomial to zero yields a set of algebraic equations for $a_{j,0}, a_{j,i}, b_{j,i}, c_{j,i}, d_{j,i}, i = 1, 2, \dots, m, j = 1, 2, \dots, k$.

Step 4. Solving the equation system in Step 3 and using (2.7), we can construct a variety of exact solutions for Eq. (3.1).

4. APPLICATION OF THE METHOD

In this section, we will construct the exact solutions of the space-time fractional ZKBBM equation by using the fractional sub-equation method.

4.1. The Space-Time Fractional ZKBBM Equation. We consider the following space-time fractional ZKBBM equation [2, 32]

$$D_t^\alpha u + D_x^\alpha u - 2auD_x^\alpha u - bD_t^\alpha (D_x^{2\alpha} u) = 0, \tag{4.1}$$



where a and b are arbitrary constants. It arises as a description of gravity water waves in the long-wave regime. Using the traveling wave transformation $u(x, t) = U(\xi)$, where $\xi = kx + ct$ and k, c are non zero constants, Eq.(4.1) can be reduced to the following nonlinear fractional ODE:

$$c^\alpha D_\xi^\alpha U + k^\alpha D_\xi^\alpha U - 2ak^\alpha U D_\xi^\alpha U - bc^\alpha k^{2\alpha} D_\xi^{3\alpha} U = 0. \quad (4.2)$$

Suppose that the solution of Eq. (4.2) can be expressed by

$$U(\xi) = a_0 + \sum_{i=1}^m \left[a_i \left(\frac{D_\xi^\alpha G}{G} \right)^i + b_i \left(\frac{D_\xi^\alpha G}{G} \right)^{i-1} \sqrt{\sigma \left\{ 1 + \frac{1}{\mu} \left(\frac{D_\xi^\alpha G}{G} \right)^2 \right\}} \right. \\ \left. + c_i \left(\frac{D_\xi^\alpha G}{G} \right)^{-i} + d_i \frac{\left(\frac{D_\xi^\alpha G}{G} \right)^{-i+1}}{\sqrt{\sigma \left\{ 1 + \frac{1}{\mu} \left(\frac{D_\xi^\alpha G}{G} \right)^2 \right\}}} \right], \quad (4.3)$$

where $G = G(\xi)$ satisfies Eq. (1.1). By balancing the order between the highest order derivative term and nonlinear term in Eq. (4.2), we can obtain $m = 2$. So, we have

$$U(\xi) = a_0 + a_1 \left(\frac{D_\xi^\alpha G}{G} \right) + a_2 \left(\frac{D_\xi^\alpha G}{G} \right)^2 + b_1 \sqrt{\sigma \left\{ 1 + \frac{1}{\mu} \left(\frac{D_\xi^\alpha G}{G} \right)^2 \right\}} \\ + b_2 \left(\frac{D_\xi^\alpha G}{G} \right) \sqrt{\sigma \left\{ 1 + \frac{1}{\mu} \left(\frac{D_\xi^\alpha G}{G} \right)^2 \right\}} + c_1 \left(\frac{D_\xi^\alpha G}{G} \right)^{-1} + c_2 \left(\frac{D_\xi^\alpha G}{G} \right)^{-2} \\ + d_1 \frac{1}{\sqrt{\sigma \left\{ 1 + \frac{1}{\mu} \left(\frac{D_\xi^\alpha G}{G} \right)^2 \right\}}} + d_2 \frac{\left(\frac{D_\xi^\alpha G}{G} \right)^{-1}}{\sqrt{\sigma \left\{ 1 + \frac{1}{\mu} \left(\frac{D_\xi^\alpha G}{G} \right)^2 \right\}}}. \quad (4.4)$$

Substituting (4.4) into (4.2) and collecting all the terms with the same power of $\left(\frac{D_\xi^\alpha G}{G} \right)$ and $\left(\frac{D_\xi^\alpha G}{G} \right) \sqrt{\sigma \left\{ 1 + \frac{1}{\mu} \left(\frac{D_\xi^\alpha G}{G} \right)^2 \right\}}$ together, equating each coefficient to zero yields a set of algebraic equations. Solving the set of algebraic equations with the help of Mathematica, we obtain the following results:

Case 1:

$$a_1 = b_1 = b_2 = c_1 = c_2 = d_1 = d_2 = 0, \\ a_0 = \frac{1 + c^\alpha (k^{-\alpha} - 8bk^\alpha \mu)}{2a}, \quad a_2 = -\frac{6bc^\alpha k^\alpha}{a}. \quad (4.5)$$



Case 2:

$$a_1 = b_1 = c_1 = c_2 = d_1 = d_2 = 0, \quad (4.6)$$

$$a_0 = \frac{1 + c^\alpha (k^{-\alpha} - 5bk^\alpha\mu)}{2a}, \quad a_2 = -\frac{3bc^\alpha k^\alpha}{a}, \quad b_2 = \pm \frac{3bc^\alpha k^\alpha \sqrt{\mu}}{a\sqrt{\sigma}}.$$

Case 3:

$$a_1 = a_2 = b_1 = b_2 = c_1 = d_1 = d_2 = 0, \quad (4.7)$$

$$a_0 = \frac{1 + c^\alpha (k^{-\alpha} - 8bk^\alpha\mu)}{2a}, \quad c_2 = -\frac{6bc^\alpha k^\alpha \mu^2}{a}.$$

Case 4:

$$a_1 = b_1 = b_2 = c_1 = d_1 = d_2 = 0, \quad (4.8)$$

$$a_0 = \frac{1 + c^\alpha (k^{-\alpha} - 8bk^\alpha\mu)}{2a}, \quad a_2 = -\frac{6bc^\alpha k^\alpha}{a}, \quad c_2 = -\frac{6bc^\alpha k^\alpha \mu^2}{a}.$$

Substituting the general solutions of Eq. (1.1) into Eq. (4.4), we can obtain the following exact solutions for Eq. (4.1).

When $\mu < 0$,

Case 1 gives

$$u(x, t) = \frac{1 + c^\alpha (k^{-\alpha} - 8bk^\alpha\mu)}{2a} + \frac{6bc^\alpha k^\alpha \mu}{a} \left[\frac{A_1 \sinh\left(\frac{\sqrt{-\mu}}{\Gamma(1+\alpha)}\xi^\alpha\right) + A_2 \cosh\left(\frac{\sqrt{-\mu}}{\Gamma(1+\alpha)}\xi^\alpha\right)}{A_1 \cosh\left(\frac{\sqrt{-\mu}}{\Gamma(1+\alpha)}\xi^\alpha\right) + A_2 \sinh\left(\frac{\sqrt{-\mu}}{\Gamma(1+\alpha)}\xi^\alpha\right)} \right]^2, \quad (4.9)$$

Case 2 gives

$$u(x, t) = \frac{1 + c^\alpha (k^{-\alpha} - 5bk^\alpha\mu)}{2a} + \frac{3bc^\alpha k^\alpha \mu}{a} \left\{ \left[\frac{A_1 \sinh\left(\frac{\sqrt{-\mu}}{\Gamma(1+\alpha)}\xi^\alpha\right) + A_2 \cosh\left(\frac{\sqrt{-\mu}}{\Gamma(1+\alpha)}\xi^\alpha\right)}{A_1 \cosh\left(\frac{\sqrt{-\mu}}{\Gamma(1+\alpha)}\xi^\alpha\right) + A_2 \sinh\left(\frac{\sqrt{-\mu}}{\Gamma(1+\alpha)}\xi^\alpha\right)} \right]^2 \right. \\ \mp \left[\frac{A_1 \sinh\left(\frac{\sqrt{-\mu}}{\Gamma(1+\alpha)}\xi^\alpha\right) + A_2 \cosh\left(\frac{\sqrt{-\mu}}{\Gamma(1+\alpha)}\xi^\alpha\right)}{A_1 \cosh\left(\frac{\sqrt{-\mu}}{\Gamma(1+\alpha)}\xi^\alpha\right) + A_2 \sinh\left(\frac{\sqrt{-\mu}}{\Gamma(1+\alpha)}\xi^\alpha\right)} \right] \\ \left. \times \sqrt{-1 + \left[\frac{A_1 \sinh\left(\frac{\sqrt{-\mu}}{\Gamma(1+\alpha)}\xi^\alpha\right) + A_2 \cosh\left(\frac{\sqrt{-\mu}}{\Gamma(1+\alpha)}\xi^\alpha\right)}{A_1 \cosh\left(\frac{\sqrt{-\mu}}{\Gamma(1+\alpha)}\xi^\alpha\right) + A_2 \sinh\left(\frac{\sqrt{-\mu}}{\Gamma(1+\alpha)}\xi^\alpha\right)} \right]^2} \right\}, \quad (4.10)$$



Case 3 gives

$$u(x, t) = \frac{1 + c^\alpha (k^{-\alpha} - 8bk^\alpha \mu)}{2a} + \frac{6bc^\alpha k^\alpha \mu}{a} \left[\frac{A_1 \cosh\left(\frac{\sqrt{-\mu}}{\Gamma(1+\alpha)} \xi^\alpha\right) + A_2 \sinh\left(\frac{\sqrt{-\mu}}{\Gamma(1+\alpha)} \xi^\alpha\right)}{A_1 \sinh\left(\frac{\sqrt{-\mu}}{\Gamma(1+\alpha)} \xi^\alpha\right) + A_2 \cosh\left(\frac{\sqrt{-\mu}}{\Gamma(1+\alpha)} \xi^\alpha\right)} \right]^2, \quad (4.11)$$

Case 4 gives

$$u(x, t) = \frac{1 + c^\alpha (k^{-\alpha} - 8bk^\alpha \mu)}{2a} + \frac{6bc^\alpha k^\alpha \mu}{a} \left\{ \left[\frac{A_1 \sinh\left(\frac{\sqrt{-\mu}}{\Gamma(1+\alpha)} \xi^\alpha\right) + A_2 \cosh\left(\frac{\sqrt{-\mu}}{\Gamma(1+\alpha)} \xi^\alpha\right)}{A_1 \cosh\left(\frac{\sqrt{-\mu}}{\Gamma(1+\alpha)} \xi^\alpha\right) + A_2 \sinh\left(\frac{\sqrt{-\mu}}{\Gamma(1+\alpha)} \xi^\alpha\right)} \right]^2 + \left[\frac{A_1 \cosh\left(\frac{\sqrt{-\mu}}{\Gamma(1+\alpha)} \xi^\alpha\right) + A_2 \sinh\left(\frac{\sqrt{-\mu}}{\Gamma(1+\alpha)} \xi^\alpha\right)}{A_1 \sinh\left(\frac{\sqrt{-\mu}}{\Gamma(1+\alpha)} \xi^\alpha\right) + A_2 \cosh\left(\frac{\sqrt{-\mu}}{\Gamma(1+\alpha)} \xi^\alpha\right)} \right]^2 \right\}, \quad (4.12)$$

where $\xi = kx + ct$.

Substituting the general solutions of Eq. (1.1) into Eq. (4.4), we can obtain the following exact solutions for Eq. (4.1).

When $\mu > 0$,

Case 1 gives

$$u(x, t) = \frac{1 + c^\alpha (k^{-\alpha} - 8bk^\alpha \mu)}{2a} - \frac{6bc^\alpha k^\alpha \mu}{a} \left[\frac{A_1 \cos\left(\frac{\sqrt{\mu}}{\Gamma(1+\alpha)} \xi^\alpha\right) - A_2 \sin\left(\frac{\sqrt{\mu}}{\Gamma(1+\alpha)} \xi^\alpha\right)}{A_1 \sin\left(\frac{\sqrt{\mu}}{\Gamma(1+\alpha)} \xi^\alpha\right) + A_2 \cos\left(\frac{\sqrt{\mu}}{\Gamma(1+\alpha)} \xi^\alpha\right)} \right]^2, \quad (4.13)$$



Case 2 gives

$$\begin{aligned}
 u(x, t) = & \frac{1 + c^\alpha (k^{-\alpha} - 5bk^\alpha \mu)}{2a} \\
 & - \frac{3bc^\alpha k^\alpha \mu}{a} \left\{ \left[\frac{A_1 \cos\left(\frac{\sqrt{\mu}}{\Gamma(1+\alpha)} \xi^\alpha\right) - A_2 \sin\left(\frac{\sqrt{\mu}}{\Gamma(1+\alpha)} \xi^\alpha\right)}{A_1 \sin\left(\frac{\sqrt{\mu}}{\Gamma(1+\alpha)} \xi^\alpha\right) + A_2 \cos\left(\frac{\sqrt{\mu}}{\Gamma(1+\alpha)} \xi^\alpha\right)} \right]^2 \right. \\
 & \mp \left[\frac{A_1 \cos\left(\frac{\sqrt{\mu}}{\Gamma(1+\alpha)} \xi^\alpha\right) - A_2 \sin\left(\frac{\sqrt{\mu}}{\Gamma(1+\alpha)} \xi^\alpha\right)}{A_1 \sin\left(\frac{\sqrt{\mu}}{\Gamma(1+\alpha)} \xi^\alpha\right) + A_2 \cos\left(\frac{\sqrt{\mu}}{\Gamma(1+\alpha)} \xi^\alpha\right)} \right] \\
 & \left. \times \sqrt{1 + \left[\frac{A_1 \cos\left(\frac{\sqrt{\mu}}{\Gamma(1+\alpha)} \xi^\alpha\right) - A_2 \sin\left(\frac{\sqrt{\mu}}{\Gamma(1+\alpha)} \xi^\alpha\right)}{A_1 \sin\left(\frac{\sqrt{\mu}}{\Gamma(1+\alpha)} \xi^\alpha\right) + A_2 \cos\left(\frac{\sqrt{\mu}}{\Gamma(1+\alpha)} \xi^\alpha\right)} \right]^2} \right\}, \tag{4.14}
 \end{aligned}$$

Case 3 gives

$$\begin{aligned}
 u(x, t) = & \frac{1 + c^\alpha (k^{-\alpha} - 8bk^\alpha \mu)}{2a} \\
 & - \frac{6bc^\alpha k^\alpha \mu}{a} \left[\frac{A_1 \sin\left(\frac{\sqrt{\mu}}{\Gamma(1+\alpha)} \xi^\alpha\right) + A_2 \cos\left(\frac{\sqrt{\mu}}{\Gamma(1+\alpha)} \xi^\alpha\right)}{A_1 \cos\left(\frac{\sqrt{\mu}}{\Gamma(1+\alpha)} \xi^\alpha\right) - A_2 \sin\left(\frac{\sqrt{\mu}}{\Gamma(1+\alpha)} \xi^\alpha\right)} \right]^2, \tag{4.15}
 \end{aligned}$$

Case 4 gives

$$\begin{aligned}
 u(x, t) = & \frac{1 + c^\alpha (k^{-\alpha} - 8bk^\alpha \mu)}{2a} \\
 & - \frac{6bc^\alpha k^\alpha \mu}{a} \left\{ \left[\frac{A_1 \cos\left(\frac{\sqrt{\mu}}{\Gamma(1+\alpha)} \xi^\alpha\right) - A_2 \sin\left(\frac{\sqrt{\mu}}{\Gamma(1+\alpha)} \xi^\alpha\right)}{A_1 \sin\left(\frac{\sqrt{\mu}}{\Gamma(1+\alpha)} \xi^\alpha\right) + A_2 \cos\left(\frac{\sqrt{\mu}}{\Gamma(1+\alpha)} \xi^\alpha\right)} \right]^2 \right. \\
 & \left. + \left[\frac{A_1 \sin\left(\frac{\sqrt{\mu}}{\Gamma(1+\alpha)} \xi^\alpha\right) + A_2 \cos\left(\frac{\sqrt{\mu}}{\Gamma(1+\alpha)} \xi^\alpha\right)}{A_1 \cos\left(\frac{\sqrt{\mu}}{\Gamma(1+\alpha)} \xi^\alpha\right) - A_2 \sin\left(\frac{\sqrt{\mu}}{\Gamma(1+\alpha)} \xi^\alpha\right)} \right]^2 \right\}, \tag{4.16}
 \end{aligned}$$

where $\xi = kx + ct$.

Substituting the general solutions of Eq. (1.1) into Eq. (4.4), we can obtain the following exact solutions for Eq. (4.1).

When $\mu = 0$,



Case 1 and 4 give

$$u(x, t) = \frac{1 + c^\alpha k^{-\alpha}}{2a} - \frac{6bc^\alpha k^\alpha}{a} \left(\frac{A_1 \Gamma(1 + \alpha)}{A_1 \xi^\alpha + A_2 \Gamma(1 + \alpha)} \right)^2, \quad (4.17)$$

Case 2 gives

$$u(x, t) = \frac{1 + c^\alpha k^{-\alpha}}{2a} - \frac{3bc^\alpha k^\alpha}{a} \left(\frac{A_1 \Gamma(1 + \alpha)}{A_1 \xi^\alpha + A_2 \Gamma(1 + \alpha)} \right)^2, \quad (4.18)$$

where $\xi = kx + ct$.

Case 3 gives

$$u(x, t) = \frac{1 + c^\alpha k^{-\alpha}}{2a}. \quad (4.19)$$

Particular cases :

Solitary, periodic and complex solutions can be derived from solutions (4.9)-(4.16) when parameters take up special values.

Solitary solutions:

(i) If $\mu < 0$, setting $A_1 = 0$, $A_2 \neq 0$ in (4.9)-(4.12), we obtain respectively the solitary wave solutions which are shown in Figure 1,

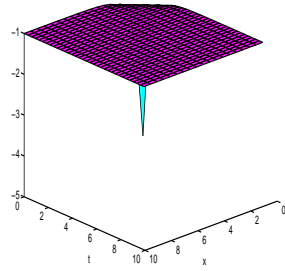
$$u(x, t) = \frac{1 + c^\alpha (k^{-\alpha} - 8bk^\alpha \mu)}{2a} + \frac{6bc^\alpha k^\alpha \mu}{a} \coth^2 \left(\frac{\sqrt{-\mu}}{\Gamma(1 + \alpha)} \xi^\alpha \right), \quad (4.20)$$

$$u(x, t) = \frac{1 + c^\alpha (k^{-\alpha} - 5bk^\alpha \mu)}{2a} + \frac{3bc^\alpha k^\alpha \mu}{a} \left\{ \coth \left(\frac{\sqrt{-\mu}}{\Gamma(1 + \alpha)} \xi^\alpha \right) \times \left[\coth \left(\frac{\sqrt{-\mu}}{\Gamma(1 + \alpha)} \xi^\alpha \right) \mp \operatorname{csch} \left(\frac{\sqrt{-\mu}}{\Gamma(1 + \alpha)} \xi^\alpha \right) \right] \right\}, \quad (4.21)$$

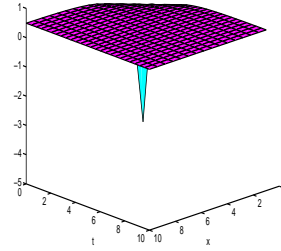
$$u(x, t) = \frac{1 + c^\alpha (k^{-\alpha} - 8bk^\alpha \mu)}{2a} + \frac{6bc^\alpha k^\alpha \mu}{a} \tanh^2 \left(\frac{\sqrt{-\mu}}{\Gamma(1 + \alpha)} \xi^\alpha \right), \quad (4.22)$$

$$u(x, t) = \frac{1 + c^\alpha (k^{-\alpha} - 8bk^\alpha \mu)}{2a} + \frac{6bc^\alpha k^\alpha \mu}{a} \left[\coth^2 \left(\frac{\sqrt{-\mu}}{\Gamma(1 + \alpha)} \xi^\alpha \right) + \tanh^2 \left(\frac{\sqrt{-\mu}}{\Gamma(1 + \alpha)} \xi^\alpha \right) \right]. \quad (4.23)$$

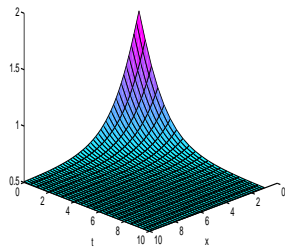




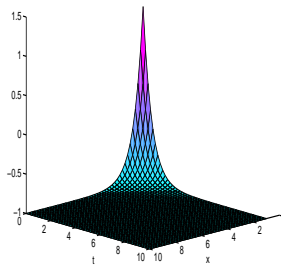
(a) Graph of $u(x, t)$ in Equation (4.20)



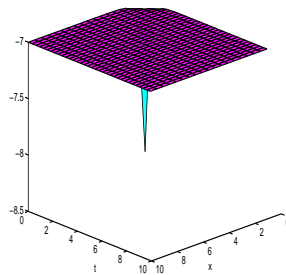
(b) Graph of $u(x, t)$ for the "+" in Equation (4.21) in place of \mp



(c) Graph of $u(x, t)$ for the "-" in Equation (4.21) in place of \mp



(d) Graph of $u(x, t)$ in Equation (4.22)



(e) Graph of $u(x, t)$ in Equation (4.23)

FIGURE 1. Profiles of the solutions (4.20)-(4.23) corresponding to the values $\alpha = 4/5$, $a = b = c = k = 1$ and $\mu = -1$.



Similarly, setting $A_1 \neq 0$, $A_2 = 0$ in (4.9) and (4.11)-(4.12), we get more solitary wave solutions which are omitted.

(ii) if $\mu < 0$ and $A_1^2 > A_2^2$ then we deduce respectively from (4.9) and (4.11)-(4.12), the solitary wave solutions,

$$u(x, t) = \frac{1 + c^\alpha (k^{-\alpha} - 8bk^\alpha \mu)}{2a} + \frac{6bc^\alpha k^\alpha \mu}{a} \tanh^2 \left(\frac{\sqrt{-\mu}}{\Gamma(1 + \alpha)} \xi^\alpha + \xi_0 \right), \quad (4.24)$$

$$u(x, t) = \frac{1 + c^\alpha (k^{-\alpha} - 8bk^\alpha \mu)}{2a} + \frac{6bc^\alpha k^\alpha \mu}{a} \coth^2 \left(\frac{\sqrt{-\mu}}{\Gamma(1 + \alpha)} \xi^\alpha + \xi_0 \right), \quad (4.25)$$

$$u(x, t) = \frac{1 + c^\alpha (k^{-\alpha} - 8bk^\alpha \mu)}{2a} + \frac{6bc^\alpha k^\alpha \mu}{a} \times \left[\tanh^2 \left(\frac{\sqrt{-\mu}}{\Gamma(1 + \alpha)} \xi^\alpha + \xi_0 \right) + \coth^2 \left(\frac{\sqrt{-\mu}}{\Gamma(1 + \alpha)} \xi^\alpha + \xi_0 \right) \right], \quad (4.26)$$

where $\xi_0 = \tanh^{-1}(A_2/A_1)$.

Periodic solutions:

(i) If $\mu > 0$, setting $A_1 = 0$, $A_2 \neq 0$ in (4.13)-(4.16), we obtain respectively the periodic wave solutions which are shown in Figure 2

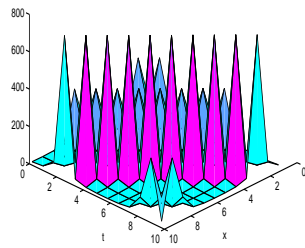
$$u(x, t) = \frac{1 + c^\alpha (k^{-\alpha} - 8bk^\alpha \mu)}{2a} - \frac{6bc^\alpha k^\alpha \mu}{a} \tan^2 \left(\frac{\sqrt{\mu}}{\Gamma(1 + \alpha)} \xi^\alpha \right), \quad (4.27)$$

$$u(x, t) = \frac{1 + c^\alpha (k^{-\alpha} - 5bk^\alpha \mu)}{2a} - \frac{3bc^\alpha k^\alpha \mu}{a} \left\{ \tan \left(\frac{\sqrt{\mu}}{\Gamma(1 + \alpha)} \xi^\alpha \right) \times \left[\tan \left(\frac{\sqrt{\mu}}{\Gamma(1 + \alpha)} \xi^\alpha \right) \pm \sec \left(\frac{\sqrt{\mu}}{\Gamma(1 + \alpha)} \xi^\alpha \right) \right] \right\}, \quad (4.28)$$

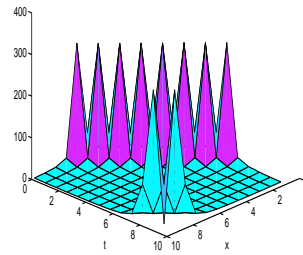
$$u(x, t) = \frac{1 + c^\alpha (k^{-\alpha} - 8bk^\alpha \mu)}{2a} - \frac{6bc^\alpha k^\alpha \mu}{a} \cot^2 \left(\frac{\sqrt{\mu}}{\Gamma(1 + \alpha)} \xi^\alpha \right), \quad (4.29)$$

$$u(x, t) = \frac{1 + c^\alpha (k^{-\alpha} - 8bk^\alpha \mu)}{2a} - \frac{6bc^\alpha k^\alpha \mu}{a} \left[\tan^2 \left(\frac{\sqrt{\mu}}{\Gamma(1 + \alpha)} \xi^\alpha \right) + \cot^2 \left(\frac{\sqrt{\mu}}{\Gamma(1 + \alpha)} \xi^\alpha \right) \right]. \quad (4.30)$$

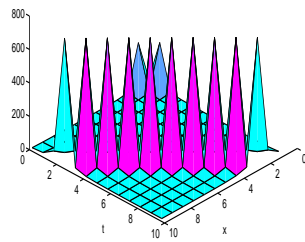




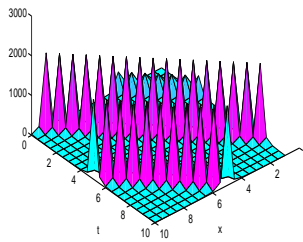
(a) Graph of $u(x,t)$ in Equation (4.27)



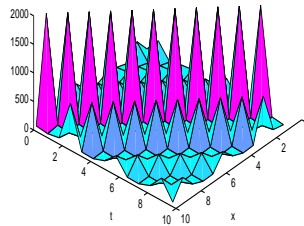
(b) Graph of $u(x,t)$ for the "-" in Equation (4.28) in place of \pm



(c) Graph of $u(x,t)$ for the "+" in Equation (4.28) in place of \pm



(d) Graph of $u(x,t)$ in Equation (4.29)



(e) Graph of $u(x,t)$ in Equation (4.30)

FIGURE 2. Profiles of the solutions (4.27)-(4.30) corresponding to the values $\alpha = 4/5$, $a = -1$, $b = c = k = 1$ and $\mu = 1$.



Similarly, setting $A_1 \neq 0$, $A_2 = 0$ in (4.13)-(4.16), we get more periodic wave solutions which are omitted.

(ii) if $\mu > 0$ we obtain respectively from (4.13)-(4.16) the periodic wave solutions,

$$u(x, t) = \frac{1 + c^\alpha (k^{-\alpha} - 8bk^\alpha \mu)}{2a} - \frac{6bc^\alpha k^\alpha \mu}{a} \tan^2 \left(\frac{\sqrt{\mu}}{\Gamma(1 + \alpha)} \xi^\alpha - \xi_0 \right), \quad (4.31)$$

$$u(x, t) = \frac{1 + c^\alpha (k^{-\alpha} - 5bk^\alpha \mu)}{2a} - \frac{3bc^\alpha k^\alpha \mu}{a} \left\{ \tan \left(\frac{\sqrt{\mu}}{\Gamma(1 + \alpha)} \xi^\alpha - \xi_0 \right) \right. \\ \left. \times \left[\tan \left(\frac{\sqrt{\mu}}{\Gamma(1 + \alpha)} \xi^\alpha - \xi_0 \right) \pm \sec \left(\frac{\sqrt{\mu}}{\Gamma(1 + \alpha)} \xi^\alpha - \xi_0 \right) \right] \right\}, \quad (4.32)$$

$$u(x, t) = \frac{1 + c^\alpha (k^{-\alpha} - 8bk^\alpha \mu)}{2a} - \frac{6bc^\alpha k^\alpha \mu}{a} \cot^2 \left(\frac{\sqrt{\mu}}{\Gamma(1 + \alpha)} \xi^\alpha - \xi_0 \right), \quad (4.33)$$

$$u(x, t) = \frac{1 + c^\alpha (k^{-\alpha} - 8bk^\alpha \mu)}{2a} - \frac{6bc^\alpha k^\alpha \mu}{a} \\ \times \left[\tan^2 \left(\frac{\sqrt{\mu}}{\Gamma(1 + \alpha)} \xi^\alpha - \xi_0 \right) + \cot^2 \left(\frac{\sqrt{\mu}}{\Gamma(1 + \alpha)} \xi^\alpha - \xi_0 \right) \right], \quad (4.34)$$

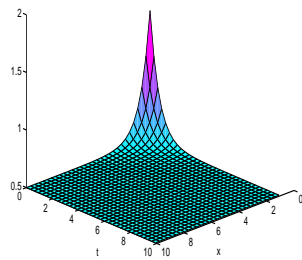
where $\xi_0 = \tan^{-1}(A_1/A_2)$.

Complex solutions:

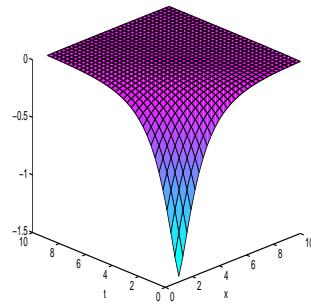
(i) If $\mu < 0$, setting $A_1 \neq 0$, $A_2 = 0$ in (4.10), we discover the complex solitary solutions which are shown in Figure 3

$$u(x, t) = \frac{1 + c^\alpha (k^{-\alpha} - 5bk^\alpha \mu)}{2a} + \frac{3bc^\alpha k^\alpha \mu}{a} \left\{ \tanh \left(\frac{\sqrt{-\mu}}{\Gamma(1 + \alpha)} \xi^\alpha \right) \right. \\ \left. \times \left[\tanh \left(\frac{\sqrt{-\mu}}{\Gamma(1 + \alpha)} \xi^\alpha \right) \mp \operatorname{isech} \left(\frac{\sqrt{-\mu}}{\Gamma(1 + \alpha)} \xi^\alpha \right) \right] \right\}. \quad (4.35)$$





(a) Graph of real $u(x, t)$ for the "+" in Equation (4.35) in place of \mp



(b) Graph of imag $u(x, t)$ for the "+" in Equation (4.35) in place of \mp

FIGURE 3. Profiles of the solutions (4.35) corresponding to the values $\alpha = 4/5$, $a = b = c = k = 1$ and $\mu = -1$.

(ii) if $\mu < 0$ and $A_1^2 > A_2^2$ then we obtain the complex solitary solutions using (4.10),

$$u(x, t) = \frac{1 + c^\alpha (k^{-\alpha} - 5bk^\alpha \mu)}{2a} + \frac{3bc^\alpha k^\alpha \mu}{a} \left\{ \tanh \left(\frac{\sqrt{-\mu}}{\Gamma(1 + \alpha)} \xi^\alpha + \xi_0 \right) \times \left[\tanh \left(\frac{\sqrt{-\mu}}{\Gamma(1 + \alpha)} \xi^\alpha + \xi_0 \right) \mp \operatorname{isech} \left(\frac{\sqrt{-\mu}}{\Gamma(1 + \alpha)} \xi^\alpha + \xi_0 \right) \right] \right\}, \quad (4.36)$$

where $\xi_0 = \tanh^{-1}(A_2/A_1)$.

To the best our knowledge, the solutions obtained in this paper have not been reported in the literature so far.

5. CONCLUSION

In this paper, we have proposed a new fractional sub-equation method for solving FPDEs with Jumarie's modified Riemann-Liouville derivative. This method is the fractional version of the known extended (G'/G) -expansion method. As an application, new exact solutions for the space-time fractional ZKBBM equation have been successfully obtained. For certain values of the parameters, solitary wave, periodic wave and complex solutions are obtained from these solutions. The method can be applied to many other FPDEs in mathematical physics.



REFERENCES

- [1] S. Abbasbandy and A. Shirzadi, *Homotopy analysis method for multiple solutions of the fractional Sturm-Liouville problems*, Numer. Algorithms, *54* (2010), 521-532.
- [2] J. F. Alzaidy, *Fractional Sub-Equation Method and its Applications to the Space-Time Fractional Differential Equations in Mathematical Physics*, British Journal of Mathematics & Computer Science, *3* (2013), 153-163.
- [3] H. Bararnla, G. Domariy, and M. Gorji, *An approximation of the analytic solution of some nonlinear heat transfer in Fin and 3D diffusion equations using HAM*, Numer. Methods Partial Differential Equations, *26* (2010), 1-13.
- [4] M. Ekici, D. Duran, and A. Sonmezoglu, *Constructing of exact solutions to the (2+1)-dimensional breaking soliton equations by the multiple (G'/G)-expansion method*, J. Adv. Math. Stud., *7* (2014), 27-44.
- [5] A. M. A. El-sayed, S. Z. Rida, and A. A. M. Arafa, *Exact solutions of fractional-order biological population model*, Commu. Theore. Physics., *52* (2009), 992-996.
- [6] F. Fouladi, E. Hosseinzad, and A. Barari, *Highly nonlinear temperature- dependent Fin analysis by variational iteration method*, Heat Transfer Res., *41* (2010), 155-165.
- [7] Z. Ganji, D. Ganji, A. D. Ganji, and M. Rostamain, *Analytical solution of time fractional Vavier-Stoke equation in polar coordinate by using homotopy analysis method*, Numer. Methods Partial Differential Equations, *26* (2010), 117-124.
- [8] K. A. Gepreel, *The homotopy perturbation method applied to the nonlinear fractional Kolmogorov-Petrovskii-Piskunov equations*, Appl. Math. Lett., *24* (2011), 1428-1434.
- [9] S. Guo and Y. Zhou, *The extended (G'/G)-expansion method and its applications to Whitham.Broer.Kaup-like equations and coupled Hirota.Satsuma KdV equations*, Appl. Math. Comput., *215* (2010), 3214-3221.
- [10] P. K. Gupta and M. Singh, *Homotopy perturbation method for fractional Fornberg-Whitham equation*, Comput. Math. Appl., *61* (2011), 250-254.
- [11] M. Hayek, *Constructing of exact solutions to the KdV and Burgers equations with power-law nonlinearity by the extended (G'/G)-expansion method*, Appl. Math. Comput., *217* (2010), 212-221.
- [12] J. H. He, *Asymptotic methods for solitary solutions and compactons*, Abst. Appl. Anal., 2012 (2012), article ID 916793, 130 pages.
- [13] M. S. Hu, R. P. Agarwal, and X. J. Yang, *Local Fractional Fourier Series with Application to Wave Equation in Fractal Vibrating String*, Abst. Appl. Anal., 2012 (2012), Article ID 567401, 15 pages.
- [14] M. Inc, *The approximate and exact solutions of the space-and time-fractional Burgers equations with initial conditions by variational iteration method*, J. Math. Anal. Appl., *345* (2008), 476-484.
- [15] A. J. M. Jawad, M. D. Petkovic, and A. Biswas, *Modified simple equation method for nonlinear evolution equations*, Appl. Math. Comput., *217* (2010), 869-877.
- [16] G. Jumarie, *Modified Riemann-Liouville derivative and fractional Taylor series of nondifferentiable functions further results*, Comput. Math. Appl., *51* (2006), 1367-1376.



- [17] G. Jumarie, *Lagrange characteristic method for solving a class of nonlinear partial differential equation of fractional order*, Appl. Math. Lett., 19 (2006), 873-880.
- [18] G. Jumarie, *Table of some basic fractional calculus formulae derived from a modified Riemann-Liouville derivative for non-differentiable functions*, Appl. Math. Lett., 22 (2009), 378-385.
- [19] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, Amsterdam, 2006.
- [20] B. Lu, *The first integral method for some time fractional differential equations*, J. Math. Anal. Appl., 395 (2012), 684-693.
- [21] W. X. Ma and B. Fuchssteiner, *Explicit and exact solutions of KPP equation*, Int. J. Nonlinear Mech., 31 (1966), 329-338.
- [22] W. X. Ma, H. Y. Wu, and J. S. He, *Partial differential equations possessing Frobenius integrable decomposition technique*, Phys. Lett. A 364 (2007), 29-32.
- [23] W. X. Ma and J. H. Lee, *A transformed rational function method and exact solution to the (3+1)-dimensional Jimbo Miwa equation*, Chaos Solitons Fractals, 42 (2009), 1356-1363.
- [24] W. X. Ma, T. Huang, and Y. Zhang, *A multiple exp-function method for nonlinear differential equations and its application*, Phys. Script., 82 (2010), 065003.
- [25] W. X. Ma and Z. Zhu, *Solving the (3+1)-dimensional generalized KP and BKP by the multiple exp-function algorithm*, Appl. Math. Comput., 218 (2012), 11871-11879.
- [26] K. S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, Wiley, New York, 1993.
- [27] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, 1999.
- [28] M. M. Rashidi, G. Domairry, A. Doosthosseini, and S. Dinarvand, *Explicit approximate solution of the coupled KdV equations by using homotopy analysis method*, Int. J. Math. Anal., 12 (2008), 581-589.
- [29] M. Safari, D. D. Ganji, and M. Moslemi, *Application of He's variational iteration and Adomian's decomposition method to the fractional KdV-Burgers-Kuramoto equation*, Comput. Math. Appl., 58 (2009), 2091-2097.
- [30] L. N. Song and H. Q. Zhang, *Solving the fractional BBM-Burgers equation using the homotopy analysis method*, Chaos, Solitons and Fractals, 40 (2009), 1616-1622.
- [31] W. H. Su, X. J. Yang, H. Jafari, and D. Baleanu, *Fractional complex transform method for wave equations on Cantor sets within local fractional differential operator*, Advances in Difference Equations, 2013(1): 97, 8 pages, doi:10.1186 /1687-1847-2013-97.
- [32] A. M. Wazwaz, *The extended tanh method for new compact and noncompact solutions for the KP-BBM and the ZK-BBM equations*, Chaos Solitons Fract., 38 (2008), 1505-1516.
- [33] G. C. Wu and E. W. M. Lee, *Fractional variational iteration method and its application*, Phys. Lett. A, 374 (2010), 2506-2509.
- [34] X. J. Yang, *Local Fractional Functional Analysis and Its Applications*, Asian Academic Publisher Limited, Hong Kong, 2011.
- [35] X. J. Yang, *Advanced Local Fractional Calculus and Its Applications*, World Science Publisher, New York, 2012.
- [36] X. J. Yang and D. Baleanu, *Fractal heat conduction problem solved by local fractional variation iteration method*, Thermal Science, 17 (2012), 625-628.



- [37] X. J. Yang, H. M. Srivastava, J. H. He, and D. Baleanu, *Cantor-type cylindrical-coordinate method for differential equations with local fractional derivatives*, Physics Letters A, *377* (2013), 1996-1700.
- [38] E. M. E. Zayed and M. A. S. EL-Malky, *The Extended (G'/G) -expansion method and its applications for solving the $(3+1)$ -dimensional nonlinear evolution equations in mathematical physics*, Global Journal of Science Frontier Research, *11* (2011), 13 pages.
- [39] E. M. E. Zayed, *A note on the modified simple equation method applied to Sharma-Tasso-Olver equation*, Appl. Math.Comput., *218* (2011), 3962-3964.
- [40] E. M. E. Zayed and S. A. Hoda Ibrahim, *Exact solutions of nonlinear evolution equations in mathematical physics using the modified simple equation method*, Chin. Phys. Lett., *29* (2012), 060201-060204.
- [41] E. M. E. Zayed and A. H. Arnous, *Exact traveling wave solutions of nonlinear PDEs in mathematical physics using the modified simple equation method*, Appl. Appl. Math., *8* (2013), 553-572.
- [42] S. Zhang and H. Q. Zhang, *Fractional sub-equation method and its application to nonlinear PDEs*, Phys. Lett. A, *375* (2011), 1069-1073.
- [43] B. Zheng, *(G'/G) -expansion method for solving fractional partial differential equations in the theory of mathematical physics*, Commu. Theor. Phys., *58* (2012), 623-630.
- [44] B. Zheng, *Exact solutions for fractional partial differential equations by a new fractional sub-equation method*, Adv. Differ. Eqs., *199* (2013), 1-11.

