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A new fractional sub-equation method for solving the space-time fractional differential equations in mathematical physics

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Abstract In this paper, a new fractional sub-equation method is proposed for finding exact solutions of fractional partial differential equations (FPDEs) in the sense of modified Riemann-Liouville derivative. With the aid of symbolic computation, we choose the space-time fractional Zakharov-Kuznetsov-Benjamin-Bona-Mahony (ZKBBM) equation in mathematical physics with a source to illustrate the validity and advantages of the novel method. As a result, some new exact solutions including solitary wave solutions and periodic wave solutions are successfully obtained. The proposed approach can also be applied to other nonlinear FPDEs arising in mathematical physics.

Keywords. Fractional sub-equation method, fractional partial differential equations, exact solutions, modified Riemann-Liouville derivative.

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1. Introduction

Nonlinear phenomena appear in a wide variety of scientific applications such as plasma physics, solid state physics and fluid dynamics. Fractional partial differential equations (FPDEs) have been attracted great interest due to their various applications in the areas of physics, biology, engineering, signal processing, control theory, finance and fractal dynamics [19, 26, 27].

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Recently, several powerful methods have been proposed to obtain approximate and exact solutions of FPDEs, such as the Adomian decomposition method [5, 29], the variational iteration method [6, 14, 33], the homotopy analysis method [1, 3, 28, 30], the homotopy perturbation method [7, 8, 10], the Lagrange characteristic method [17], the fractional sub-equation method [42], the (G'/G)-expansion method [43, 44], the first integral method [20], the transformed rational function method [23], the multiple exp-function method [24, 25], the generalized Riccati equation method [21], the Frobenius decomposition technique [22], the local fractional differential equations [34, 35], the local fractional variation iteration method [36], local fractional Fourier series method [13], the Cantor-type cylindrical coordinate method [37], the Yang-Fourier and Yang-Laplace transforms [12], the fractional complex transform method [31], the modified simple equation method [15, 39, 40, 41].

In [16], Jumarie proposed a modified Riemann-Liouville derivative. With this kind of fractional derivatives and some useful formulas, we can convert FPDEs into ordinary differential equations (ODEs) with integer orders by applying suitable transformations.

In this paper, we propose a new fractional sub-equation method to establish exact solutions for FPDEs in the sense of modified Riemann-Liouville derivative defined by Jumarie [16]. This method is a fractional version of the known extended (G'/G)-expansion method [4, 9, 11, 38]. The proposed approach is based on the following fractional ODE:

$$D_{\xi}^{2\alpha}G(\xi) + \mu G(\xi) = 0, \tag{1.1}$$

where μ is a constant and $D_{\xi}^{\alpha}G(\xi)$ denotes the modified Riemann-Liouville derivative of order α for $G(\xi)$ with respect to ξ.

The paper is arranged as follows: In Section 2, we give some definitions and properties of Jumarie's modified Riemann-Liouville derivative. We also give the expression for $\frac{D_{\xi}^{\alpha}G(\xi)}{G(\xi)}$ related to Eq. (1.1). In Section 3, we present the main steps of the fractional sub-equation method for solving FPDEs. In Section 4, we apply this method to construct exact solutions of the space-time fractional ZKBBM equation. We include figures to show the properties of some solutions of this equation. Finally, we summarize our results in the conclusion section.

2. Jumarie's modified Riemann-Liouville derivative and general expression for $\frac{D_\xi^\alpha G(\xi)}{G(\xi)}$

Jumarie's modified Riemann-Liouville derivative of order α is defined by the following expression [16]:

$$D_t^\alpha f(t) = \begin{cases} \dfrac{1}{\Gamma(1-\alpha)}\dfrac{d}{dt}\displaystyle\int_0^t (t-\xi)^{-\alpha}(f(\xi)-f(0))d\xi, & 0<\alpha<1, \\ \left(f^{(n)}(t)\right)^{(\alpha-n)}, & n\le\alpha<n+1,\ n\ge 1. \end{cases} \tag{2.1}$$

We list some important properties for the modified Riemann-Liouville derivative as follows [16]:

$$D_t^\alpha t^r = \frac{\Gamma(1+r)}{\Gamma(1+r-\alpha)}t^{r-\alpha}, \tag{2.2}$$

$$D_t^\alpha\left(f(t)g(t)\right) = g(t)D_t^\alpha f(t) + f(t)D_t^\alpha g(t), \tag{2.3}$$

$$D_t^\alpha f\left[g(t)\right] = f_g'\left[g(t)\right]D_t^\alpha g(t) = D_g^\alpha f\left[g(t)\right]\left(g'(t)\right)^\alpha. \tag{2.4}$$

In order to obtain the general solutions for Eq. (1.1), we suppose $G(\xi)=H(\eta)$ and a nonlinear fractional complex transformation $\eta=\frac{\xi^\alpha}{\Gamma(1+\alpha)}$. Then by Eq. (2.2), the first equality in Eq. (2.4) and definition of Principle of Derivative Increasing Orders [18], Eq. (1.1) can be turned into the following second ordinary differential equation

$$H''(\eta)+\mu H(\eta)=0. \tag{2.5}$$

By the general solutions of Eq. (2.5), we have

$$\frac{H'(\eta)}{H(\eta)} = \begin{cases} \sqrt{-\mu}\,\dfrac{A_1\sinh(\sqrt{-\mu}\eta)+A_2\cosh(\sqrt{-\mu}\eta)}{A_1\cosh(\sqrt{-\mu}\eta)+A_2\sinh(\sqrt{-\mu}\eta)}, & \mu<0, \\ \sqrt{\mu}\,\dfrac{A_1\cos(\sqrt{\mu}\eta)-A_2\sin(\sqrt{\mu}\eta)}{A_1\sin(\sqrt{\mu}\eta)+A_2\cos(\sqrt{\mu}\eta)}, & \mu>0, \\ \dfrac{A_1}{A_1\eta+A_2}, & \mu=0, \end{cases} \tag{2.6}$$

where A_1, A_2 are arbitrary constants.
Since $D_\xi^\alpha G(\xi) = D_\xi^\alpha H(\eta) = H'(\eta) D_\xi^\alpha \eta = H'(\eta)$, we obtain

$$\frac{D_\xi^\alpha G(\xi)}{G(\xi)} = \begin{cases} \sqrt{-\mu}\, \dfrac{A_1 \sinh\left(\frac{\sqrt{-\mu}}{\Gamma(1+\alpha)}\xi^\alpha\right) + A_2 \cosh\left(\frac{\sqrt{-\mu}}{\Gamma(1+\alpha)}\xi^\alpha\right)}{A_1 \cosh\left(\frac{\sqrt{-\mu}}{\Gamma(1+\alpha)}\xi^\alpha\right) + A_2 \sinh\left(\frac{\sqrt{-\mu}}{\Gamma(1+\alpha)}\xi^\alpha\right)}, & \mu < 0, \\ \sqrt{\mu}\, \dfrac{A_1 \cos\left(\frac{\sqrt{\mu}}{\Gamma(1+\alpha)}\xi^\alpha\right) - A_2 \sin\left(\frac{\sqrt{\mu}}{\Gamma(1+\alpha)}\xi^\alpha\right)}{A_1 \sin\left(\frac{\sqrt{\mu}}{\Gamma(1+\alpha)}\xi^\alpha\right) + A_2 \cos\left(\frac{\sqrt{\mu}}{\Gamma(1+\alpha)}\xi^\alpha\right)}, & \mu > 0, \\ \dfrac{A_1 \Gamma(1+\alpha)}{A_1 \xi^\alpha + A_2 \Gamma(1+\alpha)}, & \mu = 0. \end{cases} \tag{2.7}$$

3. Description of the fractional sub-equation method

In this section, we describe the main steps of the fractional sub-equation method for finding exact solutions of FPDEs.
Suppose that a fractional partial differential equation, say in the independent variables $t, x_1, x_2, ..., x_n$, is given by

$$\begin{aligned} P\big(&u_1, ..., u_k, D_t^\alpha u_1, ..., D_t^\alpha u_k, D_{x_1}^\alpha u_1, ..., D_{x_1}^\alpha u_k, ..., \\ &D_{x_n}^\alpha u_1, ..., D_{x_n}^\alpha u_k, D_t^{2\alpha} u_1, ..., D_t^{2\alpha} u_k, D_{x_1}^{2\alpha} u_1, ...\big) = 0, \end{aligned} \tag{3.1}$$

where $u_i = u_i(t, x_1, x_2, ..., x_n), i = 1, ..., k$ are unknown functions, P is a polynomial in u_i and their various partial derivatives including fractional derivatives.

Step 1. Suppose that

$$u_i(t, x_1, x_2, ..., x_n) = U_i(\xi), \quad \xi = ct + k_1 x_1 + k_2 x_2 + ... + k_n x_n + \xi_0. \tag{3.2}$$

Then by the second equality in Eq. (2.4), Eq. (3.1) can be turned into the following fractional ordinary differential equation with respect to the variable ξ:

$$\begin{aligned} \widetilde{P}\Big(&U_1, ..., U_k, c^\alpha D_\xi^\alpha U_1, ..., c^\alpha D_\xi^\alpha U_k, k_1^\alpha D_\xi^\alpha U_1, ..., k_1^\alpha D_\xi^\alpha U_k, ..., \\ &k_n^\alpha D_\xi^\alpha U_1, ..., k_n^\alpha D_\xi^\alpha U_k, c^{2\alpha} D_\xi^{2\alpha} U_1, ..., c^{2\alpha} D_\xi^{2\alpha} U_k, k_1^{2\alpha} D_\xi^{2\alpha} U_1, ...\Big) = 0. \end{aligned} \tag{3.3}$$

Step 2. Suppose that the solution of (3.3) can be expressed by a polynomial in $\left(D_\xi^\alpha G/G\right)$ as follows:

$$\begin{aligned} U_j(\xi) = a_{j,0} + \sum_{i=1}^{m_j} \Bigg[& a_{j,i} \left(\frac{D_\xi^\alpha G}{G}\right)^i \\ & + b_{j,i} \left(\frac{D_\xi^\alpha G}{G}\right)^{i-1} \sqrt{\sigma \left\{1 + \frac{1}{\mu}\left(\frac{D_\xi^\alpha G}{G}\right)^2\right\}} \\ & + c_{j,i} \left(\frac{D_\xi^\alpha G}{G}\right)^{-i} + d_{j,i} \frac{\left(\frac{D_\xi^\alpha G}{G}\right)^{-i+1}}{\sqrt{\sigma \left\{1 + \frac{1}{\mu}\left(\frac{D_\xi^\alpha G}{G}\right)^2\right\}}} \Bigg], \\ & j = 1, 2, ..., k, \end{aligned} \tag{3.4}$$

where $G = G(\xi)$ satisfies Eq. (1.1), σ is a constant, and $a_{j,0}, a_{j,i}, b_{j,i}, c_{j,i}, d_{j,i}$, $i = 1, 2, ..., m$, $j = 1, 2, ..., k$, are constants to be determined later. The positive integer m can be determined by considering the homogeneous balance between the highest-order derivatives and nonlinear terms appearing in (3.3).

Step 3. Substituting (3.4) along with Eq. (1.1) into Eq. (3.3) and collecting all the terms with the same order of $\left(\frac{D_\xi^\alpha G}{G}\right)$ and $\left(\frac{D_\xi^\alpha G}{G}\right)\sqrt{\sigma\left\{1 + \frac{1}{\mu}\left(\frac{D_\xi^\alpha G}{G}\right)^2\right\}}$, the left-hand side of (3.3) is converted into a polynomial in $\left(\frac{D_\xi^\alpha G}{G}\right)$ and $\left(\frac{D_\xi^\alpha G}{G}\right)\sqrt{\sigma\left\{1 + \frac{1}{\mu}\left(\frac{D_\xi^\alpha G}{G}\right)^2\right\}}$. Equating each coefficient of this polynomial to zero yields a set of algebraic equations for $a_{j,0}, a_{j,i}, b_{j,i}, c_{j,i}, d_{j,i}$, $i = 1, 2, ..., m$, $j = 1, 2, ..., k$.

Step 4. Solving the equation system in Step 3 and using (2.7), we can construct a variety of exact solutions for Eq. (3.1).

4. Application of the method

In this section, we will construct the exact solutions of the space-time fractional ZKBBM equation by using the fractional sub-equation method.

4.1. The Space-Time Fractional ZKBBM Equation.

We consider the following space-time fractional ZKBBM equation [2, 32]

$$D_t^\alpha u + D_x^\alpha u - 2auD_x^\alpha u - bD_t^\alpha\left(D_x^{2\alpha} u\right) = 0, \tag{4.1}$$

where a and b are arbitrary constants. It arises as a description of gravity water waves in the long-wave regime. Using the traveling wave transformation $u(x,t) = U(\xi)$, where $\xi = kx + ct$ and k, c are non zero constants, Eq.(4.1) can be reduced to the following nonlinear fractional ODE:

$$c^{\alpha} D_{\xi}^{\alpha} U + k^{\alpha} D_{\xi}^{\alpha} U - 2ak^{\alpha} U D_{\xi}^{\alpha} U - bc^{\alpha} k^{2\alpha} D_{\xi}^{3\alpha} U = 0. \tag{4.2}$$

Suppose that the solution of Eq. (4.2) can be expressed by

$$U(\xi) = a_0 + \sum_{i=1}^{m} \left[a_i \left(\frac{D_{\xi}^{\alpha} G}{G} \right)^{i} + b_i \left(\frac{D_{\xi}^{\alpha} G}{G} \right)^{i-1} \sqrt{\sigma \left\{ 1 + \frac{1}{\mu} \left(\frac{D_{\xi}^{\alpha} G}{G} \right)^{2} \right\}} \right.$$
$$\left. + c_i \left(\frac{D_{\xi}^{\alpha} G}{G} \right)^{-i} + d_i \frac{\left(\frac{D_{\xi}^{\alpha} G}{G} \right)^{-i+1}}{\sqrt{\sigma \left\{ 1 + \frac{1}{\mu} \left(\frac{D_{\xi}^{\alpha} G}{G} \right)^{2} \right\}}} \right], \tag{4.3}$$

where $G = G(\xi)$ satisfies Eq. (1.1). By balancing the order between the highest order derivative term and nonlinear term in Eq. (4.2), we can obtain $m = 2$. So, we have

$$U(\xi) = a_0 + a_1 \left(\frac{D_{\xi}^{\alpha} G}{G} \right) + a_2 \left(\frac{D_{\xi}^{\alpha} G}{G} \right)^{2} + b_1 \sqrt{\sigma \left\{ 1 + \frac{1}{\mu} \left(\frac{D_{\xi}^{\alpha} G}{G} \right)^{2} \right\}}$$
$$+ b_2 \left(\frac{D_{\xi}^{\alpha} G}{G} \right) \sqrt{\sigma \left\{ 1 + \frac{1}{\mu} \left(\frac{D_{\xi}^{\alpha} G}{G} \right)^{2} \right\}} + c_1 \left(\frac{D_{\xi}^{\alpha} G}{G} \right)^{-1} + c_2 \left(\frac{D_{\xi}^{\alpha} G}{G} \right)^{-2}$$
$$+ d_1 \frac{1}{\sqrt{\sigma \left\{ 1 + \frac{1}{\mu} \left(\frac{D_{\xi}^{\alpha} G}{G} \right)^{2} \right\}}} + d_2 \frac{\left(\frac{D_{\xi}^{\alpha} G}{G} \right)^{-1}}{\sqrt{\sigma \left\{ 1 + \frac{1}{\mu} \left(\frac{D_{\xi}^{\alpha} G}{G} \right)^{2} \right\}}}. \tag{4.4}$$

Substituting (4.4) into (4.2) and collecting all the terms with the same power of $\left(\frac{D_{\xi}^{\alpha} G}{G} \right)$ and $\left(\frac{D_{\xi}^{\alpha} G}{G} \right) \sqrt{\sigma \left\{ 1 + \frac{1}{\mu} \left(\frac{D_{\xi}^{\alpha} G}{G} \right)^{2} \right\}}$ together, equating each coefficient to zero yields a set of algebraic equations. Solving the set of algebraic equations with the help of Mathematica, we obtain the following results:

Case 1:

$$\begin{gathered} a_1 = b_1 = b_2 = c_1 = c_2 = d_1 = d_2 = 0, \\ a_0 = \frac{1 + c^{\alpha} \left(k^{-\alpha} - 8bk^{\alpha} \mu \right)}{2a}, \quad a_2 = -\frac{6bc^{\alpha} k^{\alpha}}{a}. \end{gathered} \tag{4.5}$$

Case 2:

$$a_1 = b_1 = c_1 = c_2 = d_1 = d_2 = 0,$$
$$a_0 = \frac{1 + c^\alpha \left(k^{-\alpha} - 5bk^\alpha\mu\right)}{2a}, \quad a_2 = -\frac{3bc^\alpha k^\alpha}{a}, \quad b_2 = \pm\frac{3bc^\alpha k^\alpha\sqrt{\mu}}{a\sqrt{\sigma}}. \tag{4.6}$$

Case 3:

$$a_1 = a_2 = b_1 = b_2 = c_1 = d_1 = d_2 = 0,$$
$$a_0 = \frac{1 + c^\alpha \left(k^{-\alpha} - 8bk^\alpha\mu\right)}{2a}, \quad c_2 = -\frac{6bc^\alpha k^\alpha \mu^2}{a}. \tag{4.7}$$

Case 4:

$$a_1 = b_1 = b_2 = c_1 = d_1 = d_2 = 0,$$
$$a_0 = \frac{1 + c^\alpha \left(k^{-\alpha} - 8bk^\alpha\mu\right)}{2a}, \quad a_2 = -\frac{6bc^\alpha k^\alpha}{a}, \quad c_2 = -\frac{6bc^\alpha k^\alpha \mu^2}{a}. \tag{4.8}$$

Substituting the general solutions of Eq. (1.1) into Eq. (4.4), we can obtain the following exact solutions for Eq. (4.1).

When $\mu < 0$,

Case 1 gives

$$u(x,t) = \frac{1 + c^\alpha \left(k^{-\alpha} - 8bk^\alpha\mu\right)}{2a} + \frac{6bc^\alpha k^\alpha \mu}{a}\left[\frac{A_1 \sinh\left(\frac{\sqrt{-\mu}}{\Gamma(1+\alpha)}\xi^\alpha\right) + A_2 \cosh\left(\frac{\sqrt{-\mu}}{\Gamma(1+\alpha)}\xi^\alpha\right)}{A_1 \cosh\left(\frac{\sqrt{-\mu}}{\Gamma(1+\alpha)}\xi^\alpha\right) + A_2 \sinh\left(\frac{\sqrt{-\mu}}{\Gamma(1+\alpha)}\xi^\alpha\right)}\right]^2, \tag{4.9}$$

Case 2 gives

$$\begin{aligned} u(x,t) = {} & \frac{1 + c^\alpha \left(k^{-\alpha} - 5bk^\alpha\mu\right)}{2a} \\ & + \frac{3bc^\alpha k^\alpha \mu}{a}\left\{\left[\frac{A_1 \sinh\left(\frac{\sqrt{-\mu}}{\Gamma(1+\alpha)}\xi^\alpha\right) + A_2 \cosh\left(\frac{\sqrt{-\mu}}{\Gamma(1+\alpha)}\xi^\alpha\right)}{A_1 \cosh\left(\frac{\sqrt{-\mu}}{\Gamma(1+\alpha)}\xi^\alpha\right) + A_2 \sinh\left(\frac{\sqrt{-\mu}}{\Gamma(1+\alpha)}\xi^\alpha\right)}\right]^2\right. \\ & \mp \left[\frac{A_1 \sinh\left(\frac{\sqrt{-\mu}}{\Gamma(1+\alpha)}\xi^\alpha\right) + A_2 \cosh\left(\frac{\sqrt{-\mu}}{\Gamma(1+\alpha)}\xi^\alpha\right)}{A_1 \cosh\left(\frac{\sqrt{-\mu}}{\Gamma(1+\alpha)}\xi^\alpha\right) + A_2 \sinh\left(\frac{\sqrt{-\mu}}{\Gamma(1+\alpha)}\xi^\alpha\right)}\right] \\ & \left.\times \sqrt{-1 + \left[\frac{A_1 \sinh\left(\frac{\sqrt{-\mu}}{\Gamma(1+\alpha)}\xi^\alpha\right) + A_2 \cosh\left(\frac{\sqrt{-\mu}}{\Gamma(1+\alpha)}\xi^\alpha\right)}{A_1 \cosh\left(\frac{\sqrt{-\mu}}{\Gamma(1+\alpha)}\xi^\alpha\right) + A_2 \sinh\left(\frac{\sqrt{-\mu}}{\Gamma(1+\alpha)}\xi^\alpha\right)}\right]^2}\right\}, \end{aligned} \tag{4.10}$$

Case 3 gives

$$u(x,t) = \frac{1 + c^{\alpha}\left(k^{-\alpha} - 8bk^{\alpha}\mu\right)}{2a} + \frac{6bc^{\alpha}k^{\alpha}\mu}{a}\left[\frac{A_1 \cosh\left(\frac{\sqrt{-\mu}}{\Gamma(1+\alpha)}\xi^{\alpha}\right) + A_2 \sinh\left(\frac{\sqrt{-\mu}}{\Gamma(1+\alpha)}\xi^{\alpha}\right)}{A_1 \sinh\left(\frac{\sqrt{-\mu}}{\Gamma(1+\alpha)}\xi^{\alpha}\right) + A_2 \cosh\left(\frac{\sqrt{-\mu}}{\Gamma(1+\alpha)}\xi^{\alpha}\right)}\right]^2, \tag{4.11}$$

Case 4 gives

$$u(x,t) = \frac{1 + c^{\alpha}\left(k^{-\alpha} - 8bk^{\alpha}\mu\right)}{2a} + \frac{6bc^{\alpha}k^{\alpha}\mu}{a}\left\{\left[\frac{A_1 \sinh\left(\frac{\sqrt{-\mu}}{\Gamma(1+\alpha)}\xi^{\alpha}\right) + A_2 \cosh\left(\frac{\sqrt{-\mu}}{\Gamma(1+\alpha)}\xi^{\alpha}\right)}{A_1 \cosh\left(\frac{\sqrt{-\mu}}{\Gamma(1+\alpha)}\xi^{\alpha}\right) + A_2 \sinh\left(\frac{\sqrt{-\mu}}{\Gamma(1+\alpha)}\xi^{\alpha}\right)}\right]^2 + \left[\frac{A_1 \cosh\left(\frac{\sqrt{-\mu}}{\Gamma(1+\alpha)}\xi^{\alpha}\right) + A_2 \sinh\left(\frac{\sqrt{-\mu}}{\Gamma(1+\alpha)}\xi^{\alpha}\right)}{A_1 \sinh\left(\frac{\sqrt{-\mu}}{\Gamma(1+\alpha)}\xi^{\alpha}\right) + A_2 \cosh\left(\frac{\sqrt{-\mu}}{\Gamma(1+\alpha)}\xi^{\alpha}\right)}\right]^2\right\}, \tag{4.12}$$

where $\xi = kx + ct$.
Substituting the general solutions of Eq. (1.1) into Eq. (4.4), we can obtain the following exact solutions for Eq. (4.1).

When $\mu > 0$,

Case 1 gives

$$u(x,t) = \frac{1 + c^{\alpha}\left(k^{-\alpha} - 8bk^{\alpha}\mu\right)}{2a} - \frac{6bc^{\alpha}k^{\alpha}\mu}{a}\left[\frac{A_1 \cos\left(\frac{\sqrt{\mu}}{\Gamma(1+\alpha)}\xi^{\alpha}\right) - A_2 \sin\left(\frac{\sqrt{\mu}}{\Gamma(1+\alpha)}\xi^{\alpha}\right)}{A_1 \sin\left(\frac{\sqrt{\mu}}{\Gamma(1+\alpha)}\xi^{\alpha}\right) + A_2 \cos\left(\frac{\sqrt{\mu}}{\Gamma(1+\alpha)}\xi^{\alpha}\right)}\right]^2, \tag{4.13}$$

Case 2 gives

$$\begin{aligned} u(x,t) = & \frac{1 + c^\alpha \left(k^{-\alpha} - 5bk^\alpha \mu\right)}{2a} \\ & - \frac{3bc^\alpha k^\alpha \mu}{a} \left\{ \left[\frac{A_1 \cos\left(\frac{\sqrt{\mu}}{\Gamma(1+\alpha)}\xi^\alpha\right) - A_2 \sin\left(\frac{\sqrt{\mu}}{\Gamma(1+\alpha)}\xi^\alpha\right)}{A_1 \sin\left(\frac{\sqrt{\mu}}{\Gamma(1+\alpha)}\xi^\alpha\right) + A_2 \cos\left(\frac{\sqrt{\mu}}{\Gamma(1+\alpha)}\xi^\alpha\right)} \right]^2 \right. \\ & \mp \left[\frac{A_1 \cos\left(\frac{\sqrt{\mu}}{\Gamma(1+\alpha)}\xi^\alpha\right) - A_2 \sin\left(\frac{\sqrt{\mu}}{\Gamma(1+\alpha)}\xi^\alpha\right)}{A_1 \sin\left(\frac{\sqrt{\mu}}{\Gamma(1+\alpha)}\xi^\alpha\right) + A_2 \cos\left(\frac{\sqrt{\mu}}{\Gamma(1+\alpha)}\xi^\alpha\right)} \right] \\ & \left. \times \sqrt{1 + \left[\frac{A_1 \cos\left(\frac{\sqrt{\mu}}{\Gamma(1+\alpha)}\xi^\alpha\right) - A_2 \sin\left(\frac{\sqrt{\mu}}{\Gamma(1+\alpha)}\xi^\alpha\right)}{A_1 \sin\left(\frac{\sqrt{\mu}}{\Gamma(1+\alpha)}\xi^\alpha\right) + A_2 \cos\left(\frac{\sqrt{\mu}}{\Gamma(1+\alpha)}\xi^\alpha\right)} \right]^2} \right\}, \end{aligned} \tag{4.14}$$

Case 3 gives

$$\begin{aligned} u(x,t) = & \frac{1 + c^\alpha \left(k^{-\alpha} - 8bk^\alpha \mu\right)}{2a} \\ & - \frac{6bc^\alpha k^\alpha \mu}{a} \left[\frac{A_1 \sin\left(\frac{\sqrt{\mu}}{\Gamma(1+\alpha)}\xi^\alpha\right) + A_2 \cos\left(\frac{\sqrt{\mu}}{\Gamma(1+\alpha)}\xi^\alpha\right)}{A_1 \cos\left(\frac{\sqrt{\mu}}{\Gamma(1+\alpha)}\xi^\alpha\right) - A_2 \sin\left(\frac{\sqrt{\mu}}{\Gamma(1+\alpha)}\xi^\alpha\right)} \right]^2, \end{aligned} \tag{4.15}$$

Case 4 gives

$$\begin{aligned} u(x,t) = & \frac{1 + c^\alpha \left(k^{-\alpha} - 8bk^\alpha \mu\right)}{2a} \\ & - \frac{6bc^\alpha k^\alpha \mu}{a} \left\{ \left[\frac{A_1 \cos\left(\frac{\sqrt{\mu}}{\Gamma(1+\alpha)}\xi^\alpha\right) - A_2 \sin\left(\frac{\sqrt{\mu}}{\Gamma(1+\alpha)}\xi^\alpha\right)}{A_1 \sin\left(\frac{\sqrt{\mu}}{\Gamma(1+\alpha)}\xi^\alpha\right) + A_2 \cos\left(\frac{\sqrt{\mu}}{\Gamma(1+\alpha)}\xi^\alpha\right)} \right]^2 \right. \\ & \left. + \left[\frac{A_1 \sin\left(\frac{\sqrt{\mu}}{\Gamma(1+\alpha)}\xi^\alpha\right) + A_2 \cos\left(\frac{\sqrt{\mu}}{\Gamma(1+\alpha)}\xi^\alpha\right)}{A_1 \cos\left(\frac{\sqrt{\mu}}{\Gamma(1+\alpha)}\xi^\alpha\right) - A_2 \sin\left(\frac{\sqrt{\mu}}{\Gamma(1+\alpha)}\xi^\alpha\right)} \right]^2 \right\}, \end{aligned} \tag{4.16}$$

where $\xi = kx + ct$.

Substituting the general solutions of Eq. (1.1) into Eq. (4.4), we can obtain the following exact solutions for Eq. (4.1).

When $\mu = 0$,

Case 1 and 4 give

$$u(x,t) = \frac{1 + c^{\alpha}k^{-\alpha}}{2a} - \frac{6bc^{\alpha}k^{\alpha}}{a}\left(\frac{A_1\Gamma(1+\alpha)}{A_1\xi^{\alpha} + A_2\Gamma(1+\alpha)}\right)^2, \tag{4.17}$$

Case 2 gives

$$u(x,t) = \frac{1 + c^{\alpha}k^{-\alpha}}{2a} - \frac{3bc^{\alpha}k^{\alpha}}{a}\left(\frac{A_1\Gamma(1+\alpha)}{A_1\xi^{\alpha} + A_2\Gamma(1+\alpha)}\right)^2, \tag{4.18}$$

where $\xi = kx + ct$.
Case 3 gives

$$u(x,t) = \frac{1 + c^{\alpha}k^{-\alpha}}{2a}. \tag{4.19}$$

Particular cases :
Solitary, periodic and complex solutions can be derived from solutions (4.9)-(4.16) when parameters take up special values.

Solitary solutions:

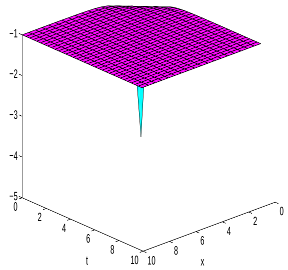
(i) If $\mu < 0$, setting $A_1 = 0$, $A_2 \neq 0$ in (4.9)-(4.12), we obtain respectively the solitary wave solutions which are shown in Figure 1,

$$u(x,t) = \frac{1 + c^{\alpha}\left(k^{-\alpha} - 8bk^{\alpha}\mu\right)}{2a} + \frac{6bc^{\alpha}k^{\alpha}\mu}{a}\coth^2\left(\frac{\sqrt{-\mu}}{\Gamma(1+\alpha)}\xi^{\alpha}\right), \tag{4.20}$$

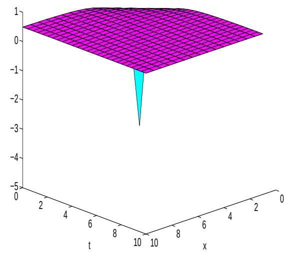
$$\begin{aligned} u(x,t) = &\frac{1 + c^{\alpha}\left(k^{-\alpha} - 5bk^{\alpha}\mu\right)}{2a} + \frac{3bc^{\alpha}k^{\alpha}\mu}{a}\left\{\coth\left(\frac{\sqrt{-\mu}}{\Gamma(1+\alpha)}\xi^{\alpha}\right)\right.\\ &\left.\times\left[\coth\left(\frac{\sqrt{-\mu}}{\Gamma(1+\alpha)}\xi^{\alpha}\right) \mp \operatorname{csch}\left(\frac{\sqrt{-\mu}}{\Gamma(1+\alpha)}\xi^{\alpha}\right)\right]\right\}, \end{aligned} \tag{4.21}$$

$$u(x,t) = \frac{1 + c^{\alpha}\left(k^{-\alpha} - 8bk^{\alpha}\mu\right)}{2a} + \frac{6bc^{\alpha}k^{\alpha}\mu}{a}\tanh^2\left(\frac{\sqrt{-\mu}}{\Gamma(1+\alpha)}\xi^{\alpha}\right), \tag{4.22}$$

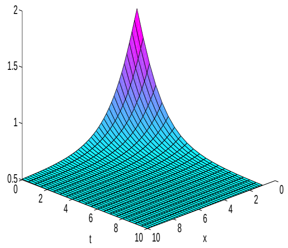
$$\begin{aligned} u(x,t) = &\frac{1 + c^{\alpha}\left(k^{-\alpha} - 8bk^{\alpha}\mu\right)}{2a}\\ &+ \frac{6bc^{\alpha}k^{\alpha}\mu}{a}\left[\coth^2\left(\frac{\sqrt{-\mu}}{\Gamma(1+\alpha)}\xi^{\alpha}\right) + \tanh^2\left(\frac{\sqrt{-\mu}}{\Gamma(1+\alpha)}\xi^{\alpha}\right)\right]. \end{aligned} \tag{4.23}$$



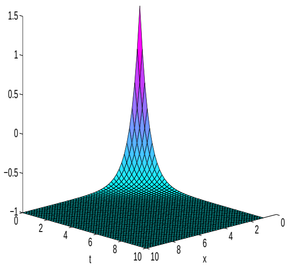
(a) Graph of $u(x,t)$ in Equation (4.20)



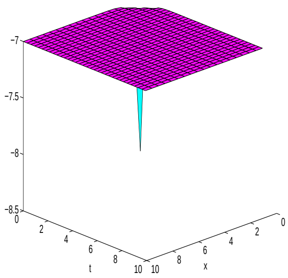
(b) Graph of $u(x,t)$ for the "+" in Equation (4.21) in place of $\mp$



(c) Graph of $u(x,t)$ for the "-" in Equation (4.21) in place of $\mp$



(d) Graph of $u(x,t)$ in Equation (4.22)



(e) Graph of $u(x,t)$ in Equation (4.23)

FIGURE 1. Profiles of the solutions (4.20)-(4.23) corresponding to the values $\alpha = 4/5,\ a = b = c = k = 1$ and $\mu = -1$.

Similarly, setting $A_1 \neq 0,\ A_2 = 0$ in (4.9) and (4.11)-(4.12), we get more solitary wave solutions which are omitted.
(ii) if $\mu < 0$ and $A_1^2 > A_2^2$ then we deduce respectively from (4.9) and (4.11)-(4.12), the solitary wave solutions,

$$u(x,t) = \frac{1 + c^{\alpha}\left(k^{-\alpha} - 8bk^{\alpha}\mu\right)}{2a} + \frac{6bc^{\alpha}k^{\alpha}\mu}{a}\tanh^2\left(\frac{\sqrt{-\mu}}{\Gamma(1+\alpha)}\xi^{\alpha} + \xi_0\right), \quad (4.24)$$

$$u(x,t) = \frac{1 + c^{\alpha}\left(k^{-\alpha} - 8bk^{\alpha}\mu\right)}{2a} + \frac{6bc^{\alpha}k^{\alpha}\mu}{a}\coth^2\left(\frac{\sqrt{-\mu}}{\Gamma(1+\alpha)}\xi^{\alpha} + \xi_0\right), \quad (4.25)$$

$$u(x,t) = \frac{1 + c^{\alpha}\left(k^{-\alpha} - 8bk^{\alpha}\mu\right)}{2a} + \frac{6bc^{\alpha}k^{\alpha}\mu}{a} \times\left[\tanh^2\left(\frac{\sqrt{-\mu}}{\Gamma(1+\alpha)}\xi^{\alpha} + \xi_0\right) + \coth^2\left(\frac{\sqrt{-\mu}}{\Gamma(1+\alpha)}\xi^{\alpha} + \xi_0\right)\right], \quad (4.26)$$

where $\xi_0 = \tanh^{-1}(A_2/A_1)$.

Periodic solutions:

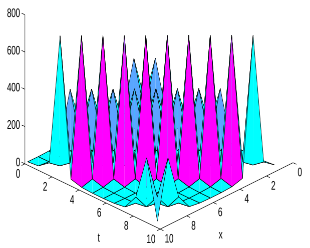
(i) If $\mu > 0$, setting $A_1 = 0,\ A_2 \neq 0$ in (4.13)-(4.16), we obtain respectively the periodic wave solutions which are shown in Figure 2

$$u(x,t) = \frac{1 + c^{\alpha}\left(k^{-\alpha} - 8bk^{\alpha}\mu\right)}{2a} - \frac{6bc^{\alpha}k^{\alpha}\mu}{a}\tan^2\left(\frac{\sqrt{\mu}}{\Gamma(1+\alpha)}\xi^{\alpha}\right), \quad (4.27)$$

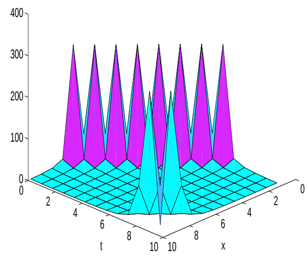
$$u(x,t) = \frac{1 + c^{\alpha}\left(k^{-\alpha} - 5bk^{\alpha}\mu\right)}{2a} - \frac{3bc^{\alpha}k^{\alpha}\mu}{a}\left\{\tan\left(\frac{\sqrt{\mu}}{\Gamma(1+\alpha)}\xi^{\alpha}\right) \times\left[\tan\left(\frac{\sqrt{\mu}}{\Gamma(1+\alpha)}\xi^{\alpha}\right) \pm \sec\left(\frac{\sqrt{\mu}}{\Gamma(1+\alpha)}\xi^{\alpha}\right)\right]\right\}, \quad (4.28)$$

$$u(x,t) = \frac{1 + c^{\alpha}\left(k^{-\alpha} - 8bk^{\alpha}\mu\right)}{2a} - \frac{6bc^{\alpha}k^{\alpha}\mu}{a}\cot^2\left(\frac{\sqrt{\mu}}{\Gamma(1+\alpha)}\xi^{\alpha}\right), \quad (4.29)$$

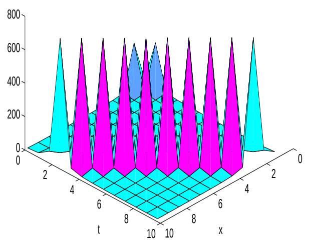
$$u(x,t) = \frac{1 + c^{\alpha}\left(k^{-\alpha} - 8bk^{\alpha}\mu\right)}{2a} - \frac{6bc^{\alpha}k^{\alpha}\mu}{a}\left[\tan^2\left(\frac{\sqrt{\mu}}{\Gamma(1+\alpha)}\xi^{\alpha}\right) + \cot^2\left(\frac{\sqrt{\mu}}{\Gamma(1+\alpha)}\xi^{\alpha}\right)\right]. \quad (4.30)$$



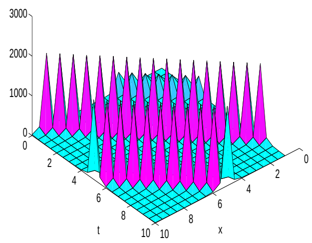
(a) Graph of $u(x,t)$ in Equation (4.27)



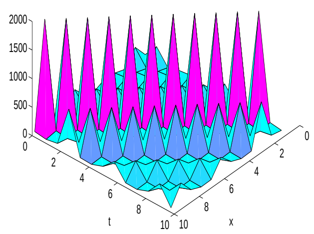
(b) Graph of $u(x,t)$ for the "-" in Equation (4.28) in place of $\pm$



(c) Graph of $u(x,t)$ for the "+" in Equation (4.28) in place of $\pm$



(d) Graph of $u(x,t)$ in Equation (4.29)



(e) Graph of $u(x,t)$ in Equation (4.30)

FIGURE 2. Profiles of the solutions (4.27)-(4.30) corresponding to the values $\alpha = 4/5,\ a = -1,\ b = c = k = 1$ and $\mu = 1$.

Similarly, setting $A_1 \neq 0$, $A_2 = 0$ in (4.13)-(4.16), we get more periodic wave solutions which are omitted.
(ii) if $\mu > 0$ we obtain respectively from (4.13)-(4.16) the periodic wave solutions,

$$u(x,t) = \frac{1 + c^\alpha \left(k^{-\alpha} - 8bk^\alpha \mu\right)}{2a} - \frac{6bc^\alpha k^\alpha \mu}{a} \tan^2\left(\frac{\sqrt{\mu}}{\Gamma(1+\alpha)}\xi^\alpha - \xi_0\right), \quad (4.31)$$

$$\begin{aligned} u(x,t) = {} & \frac{1 + c^\alpha \left(k^{-\alpha} - 5bk^\alpha \mu\right)}{2a} - \frac{3bc^\alpha k^\alpha \mu}{a} \left\{\tan\left(\frac{\sqrt{\mu}}{\Gamma(1+\alpha)}\xi^\alpha - \xi_0\right)\right. \\ & \left. \times \left[\tan\left(\frac{\sqrt{\mu}}{\Gamma(1+\alpha)}\xi^\alpha - \xi_0\right) \pm \sec\left(\frac{\sqrt{\mu}}{\Gamma(1+\alpha)}\xi^\alpha - \xi_0\right)\right]\right\}, \end{aligned} \quad (4.32)$$

$$u(x,t) = \frac{1 + c^\alpha \left(k^{-\alpha} - 8bk^\alpha \mu\right)}{2a} - \frac{6bc^\alpha k^\alpha \mu}{a} \cot^2\left(\frac{\sqrt{\mu}}{\Gamma(1+\alpha)}\xi^\alpha - \xi_0\right), \quad (4.33)$$

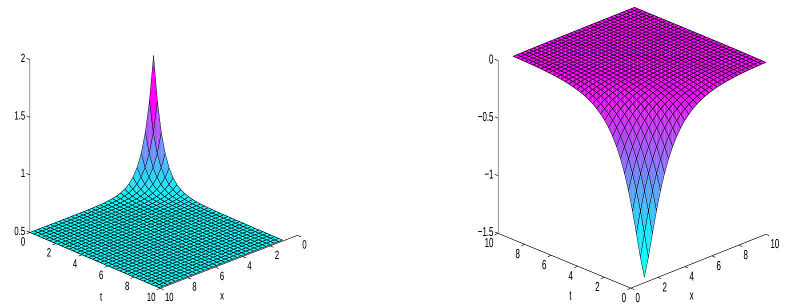
$$\begin{aligned} u(x,t) = {} & \frac{1 + c^\alpha \left(k^{-\alpha} - 8bk^\alpha \mu\right)}{2a} - \frac{6bc^\alpha k^\alpha \mu}{a} \\ & \times \left[\tan^2\left(\frac{\sqrt{\mu}}{\Gamma(1+\alpha)}\xi^\alpha - \xi_0\right) + \cot^2\left(\frac{\sqrt{\mu}}{\Gamma(1+\alpha)}\xi^\alpha - \xi_0\right)\right], \end{aligned} \quad (4.34)$$

where $\xi_0 = \tan^{-1}(A_1/A_2)$.

Complex solutions:

(i) If $\mu < 0$, setting $A_1 \neq 0$, $A_2 = 0$ in (4.10), we discover the complex solitary solutions which are shown in Figure 3

$$\begin{aligned} u(x,t) = {} & \frac{1 + c^\alpha \left(k^{-\alpha} - 5bk^\alpha \mu\right)}{2a} + \frac{3bc^\alpha k^\alpha \mu}{a} \left\{\tanh\left(\frac{\sqrt{-\mu}}{\Gamma(1+\alpha)}\xi^\alpha\right)\right. \\ & \left. \times \left[\tanh\left(\frac{\sqrt{-\mu}}{\Gamma(1+\alpha)}\xi^\alpha\right) \mp i\operatorname{sech}\left(\frac{\sqrt{-\mu}}{\Gamma(1+\alpha)}\xi^\alpha\right)\right]\right\}. \end{aligned} \quad (4.35)$$



(a) Graph of real $u(x,t)$ for the "+" in Equation (4.35) in place of $\mp$

(b) Graph of imag $u(x,t)$ for the "+" in Equation (4.35) in place of $\mp$

FIGURE 3. Profiles of the solutions (4.35) corresponding to the values $\alpha = 4/5,\ a = b = c = k = 1$ and $\mu = -1$.

(ii) if $\mu < 0$ and $A_1^2 > A_2^2$ then we obtain the complex solitary solutions using (4.10),

$$u(x,t) = \frac{1 + c^{\alpha}\left(k^{-\alpha} - 5bk^{\alpha}\mu\right)}{2a} + \frac{3bc^{\alpha}k^{\alpha}\mu}{a}\left\{\tanh\left(\frac{\sqrt{-\mu}}{\Gamma(1+\alpha)}\xi^{\alpha} + \xi_0\right)\right.$$
$$\left.\times\left[\tanh\left(\frac{\sqrt{-\mu}}{\Gamma(1+\alpha)}\xi^{\alpha} + \xi_0\right) \mp i\,\mathrm{sech}\left(\frac{\sqrt{-\mu}}{\Gamma(1+\alpha)}\xi^{\alpha} + \xi_0\right)\right]\right\}, \quad (4.36)$$

where $\xi_0 = \tanh^{-1}(A_2/A_1)$.

To the best our knowledge, the solutions obtained in this paper have not been reported in the literature so far.

5. Conclusion

In this paper, we have proposed a new fractional sub-equation method for solving FPDEs with Jumarie's modified Riemann-Liouville derivative. This method is the fractional version of the known extended (G'/G)-expansion method. As an application, new exact solutions for the space-time fractional ZKBBM equation have been successfully obtained. For certain values of the parameters, solitary wave, periodic wave and complex solutions are obtained from these solutions. The method can be applied to many other FPDEs in mathematical physics.

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