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Existence and uniqueness of solutions for p-Laplacian fractional order boundary value problems

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Abstract In this paper, we study sufficient conditions for existence and uniqueness of solutions of three point boundary value problem for p-Laplacian fractional order differential equations. We use Schauder's fixed point theorem for existence of solutions and concavity of the operator for uniqueness of solution. We include some examples to show the applicability of our results.

Keywords. Fractional differential equations, Three point boundary conditions, Fixed point theorems, p-Laplacian operator.

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1. Introduction

The rapidly increasing applications of fractional order differential equations in various fields of sciences such as Engineering, Mathematics, Chemistry, etc [10, 11, 15–17, 21], attracted the interest of many modern scientists. One of the most important area of research in the field of fractional order differential equations is the theory on existence and uniqueness of solutions to nonlinear boundary value problems for fractional order differential equations. This ares of research gained much interest in the community of mathematicians and is rapidly growing area. We refer the readers to the recent work [1–7, 12–14, 18–20, 22, 25] and the references therein for the valuable results on the theory of existence of solutions to boundary value problems corresponding to fractional order differential equations.

The theory on existence and uniqueness of solutions to boundary value problems with p-Laplacian operator for ordinary differential equations are well studied. For example, J. Zhang et.al [26] studied multiple periodic solutions of p-Laplacian equation of the form

$$\begin{cases} (\phi_p(u'))' = f(t, u, u'),\ t \in [0, T], \\ u(0) = u(T), \quad u'(0) = u'(T), \end{cases} \tag{1.1}$$

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with the tools of degree theory and the method of upper and lower solutions. X. Xu and B. Xu [24] studied sign changing solutions of p-Laplacian equation with a sub-linear nonlinearity at infinity

$$\begin{cases} (\phi_p(u'(t)))' + f(t, u(t), u'(t)) = 0, \quad t \in (0, 1), \\ u(0) = u(1) = 0, \end{cases} \tag{1.2}$$

by the use of upper and lower solutions method and Leray-Schauder degree theory. In [23], B. Wang studied triple positive solutions for boundary value problems for one-dimensional p-Laplacian on a half line of the form

$$\begin{cases} (\phi_p(x'(t)))' + h(t)f(t, x(t), x'(t)) = 0, \ 0 < t < \infty, \\ u(0) = 0, \quad \lim_{t \to +\infty} x(t) = 0. \end{cases} \tag{1.3}$$

The study of boundary value problems, particulary multi point boundary value problems for fractional differential equations with p-Laplacian operators has attracted the attentions of mathematicians quite recently and only few paper can be found in the literature dealing with p-Laplacian fractional order boundary value problems. Z. Han et.al [9] studied positive solutions to boundary value problems of p-Laplacian fractional differential equations of the form

$$\begin{cases} D^{\beta}_{0+}(\phi_p(D^{\alpha}u(t))) + a(t)f(u(t)) = 0, \ 0 < t < 1 \\ u(0) = \gamma(\xi) + \lambda, \ \phi_p(D^{\alpha}_{0+}u(0)) = (\phi_p(D^{\alpha}_{0+}u(1)))' = (\phi_p(D^{\alpha}_{0+}u(0)))'' = 0, \end{cases} \tag{1.4}$$

where $0 < \alpha \leq 1$, $2 < \beta \leq 3$ are real numbers and D^{α}_{0+}, D^{β}_{0+} are standard Caputo fractional fractional derivatives.

Motivated by the above work, we studied existence and uniqueness of solutions to three point boundary value problems for p-Laplacian fractional order differential equation of the form

$$\begin{cases} D^{\alpha}(\phi_p(D^{\beta}u(t))) + a(t)f(u(t)) = 0, \ t \in [0, 1], \ 2 < \alpha, \beta \leq 3, \\ u(0) = 0, \ \gamma u'(1) = u'(0), \ u''(0) = 0 \\ \phi_p(D^{\beta}u(0)) = 0, \phi_p(D^{\beta}u(\xi)) = (\phi_p(D^{\beta}u(1)))', (\phi_p(D^{\beta}u(0)))'' = 0, \end{cases} \tag{1.5}$$

where $0 < \xi, \gamma < 1$, D^{α}, D^{β} stand for Caputo's fractional derivative and $\phi_p(s) = |s|^{p-2}s$, $p > 1$, $\phi_p^{-1} = \phi_q$, $\frac{1}{p} + \frac{1}{q} = 1$. We recall some basic definitions and results. For $\alpha > 0$, choose $n = [\alpha] + 1$ if α in not an integer and $n = \alpha$ if α is an integer.

Definition 1.1. The fractional order integral of order $\alpha > 0$ of a function $f : (0, \infty) \to R$ is given by

$$I^{\alpha}f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} f(s)\, ds,$$

provided the integral converges.

Definition 1.2. For a function $f \in C^n[0,1]$, the Caputo fractional derivative of order α is define by

$$(D^{\alpha})f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s)ds,$$

provided that the right side is pointwise defined on $(0,\infty)$.

Definition 1.3. A cone P in a real Banach space X is called solid if $P^o \neq \emptyset$, where P^o is the interior of P. A cone P of a real Banach space X, is normal if there exists $N > 0$ such that $x \leq y$ implies that $\|x\| \leq N\|y\|$ for each x, $y \in P$, and the minimal N is called a normal constant of P.

Definition 1.4. Let P be a solid cone in a real Banach space X, $T : P^o \to P^o$ be an operator and $0 < \theta < 1$. Then T is called θ-concave operator if $T(ku) \geq k^{\theta} T(u)$ for any $0 < k < 1$ and $u \in P^o$.

Lemma 1.5 ([8]). *Assume that P is a normal solid cone in a real Banach space X, $0 < \theta < 1$ and $T : P^o \to P^o$ is a $\theta-$concave increasing operator. Then T has only one fixed point in P^o.*

The following results are known [11].

Lemma 1.6. *For $\alpha, \beta > 0$, the following relation hold:*

$$D^{\alpha} t^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} t^{\beta-\alpha-1}, \beta > n \text{ and } D^{\alpha} t^k = 0, k = 0, 1, 2, ..., n-1.$$

Lemma 1.7. *For $g(t) \in C(0,1)$, the homogenous fractional order differential equation $D^{\alpha} g(t) = 0$ has a solution*

$$g(t) = c_1 + c_2 t + c_3 t^2 + ... + c_n t^{n-1},\ c_i \in R, i = 1, 2, 3, ..., n. \tag{1.6}$$

Lemma 1.8. *The following result holds for fractional differential equations*

$$I^{\alpha} D^{\alpha} y(t) = y(t) + c_0 + c_1 t + c_2 t^2 + ... + c_{n-1} t^{n-1},$$

for arbitrary $c_i \in R$, $\quad i = 0, 1, 2, ..., n-1$.

2. MAIN RESULTS

We need the following lemmas for the proof of our main results.

Lemma 2.1. *For $y \in C[0,1]$, the boundary value problem for fractional differential equation*

$$\begin{cases} D^{\beta} u(t) = y(t)\, 2 < \beta \leq 3, \\ u(0) = 0, \quad \gamma u'(1) = u'(0), \quad u''(0) = 0, \end{cases} \tag{2.1}$$

has a solution of the form

$$u(t) = \int_0^1 G(t,s) y(s) ds, \tag{2.2}$$

where

$$G(t,s)=\begin{cases}\frac{1}{\Gamma(\beta)}(t-s)^{\beta-1}+\frac{t}{1-\gamma}\frac{\gamma}{\Gamma(\beta-1)}(1-s)^{\beta-2}, & 0<s\le t<1\\ \frac{t}{1-\gamma}\frac{\gamma}{\Gamma(\beta-1)}(1-s)^{\beta-2}, & 0<t\le s<1.\end{cases} \tag{2.3}$$

Proof. Applying the operator I^{β} on the differential equation in (2.1) and using lemma (1.8), we obtain

$$u(t)=I^{\beta}y(t)+c_1+c_2t+c_3t^2. \tag{2.4}$$

The boundary conditions $u(0)=0$ and $u''(0)=0$ imply that $c_1=0=c_3$, and the boundary condition $\gamma u'(1)=u'(0)$ yields $c_2=\frac{\gamma}{1-\gamma}I^{\beta-1}y(1)$. Hence, (2.4) takes the form

$$u(t)=I^{\beta}y(t)+\frac{t\gamma}{1-\gamma}I^{\beta-1}y(1), \tag{2.5}$$

which can be rewritten as

$$\begin{aligned}u(t)&=\frac{1}{\Gamma(\beta)}\int_0^t(t-s)^{\beta-1}y(s)ds+\frac{t\gamma}{1-\gamma}\frac{1}{\Gamma(\beta-1)}\int_0^1(1-s)^{\beta-2}y(s)ds\\&=\int_0^1 G(t,s)y(s)ds.\end{aligned}$$

□

We note that $G(t,s)\ge 0$ on $[0,1]\times[0,1]$. Further, for $t_1,\ t_2\in[0,1]$ with $s\le t_1\le t_2$, we have

$$\begin{aligned}G(t_2,s)-G(t_1,s)&=\frac{1}{\Gamma(\beta)}((t_2-s)^{\beta-1}-(t_1-s)^{\beta-1})\\&\qquad+\frac{\gamma(1-s)^{\beta-2}}{(1-\gamma)\Gamma(\beta-1)}(t_2-t_1)\\&\le\left(\frac{(\beta-1)c^{\beta-2}}{\Gamma(\beta)}+\frac{\gamma}{(1-\gamma)\Gamma(\beta-1)}\right)(t_2-t_1),\end{aligned} \tag{2.6}$$

$c\in(t_1,t_2)$ and for $t_1,\ t_2\in[0,1]$ with $t_1\le t_2\le s$, we have

$$G(t_2,s)-G(t_1,s)=\frac{\gamma(1-s^{\beta-2})}{(1-\gamma)\Gamma(\beta-1)}(t_2-t_1)\le\frac{\gamma}{(1-\gamma)\Gamma(\beta-1)})(t_2-t_1). \tag{2.7}$$

From (2.6) and (2.7), it follows that

$$G(t_2,s)-G(t_1,s)\to 0 \text{ as } t_1\to t_2. \tag{2.8}$$

Lemma 2.2. *For $y\in C[0,1]$, the boundary value problem for fractional differential equation*

$$\begin{cases}D^{\alpha}(\phi_p(D^{\beta}u(t)))+y(t)=0, \quad 2<\alpha,\beta\le 3,\\ u(0)=0,\ \gamma u'(1)=u'(0),\ u''(0)=0,\\ \phi_p(D^{\beta}u(0))=0,\ \phi_p(D^{\beta}u(\xi))=(\phi_p(D^{\beta}u(1)))',\ (\phi_p(D^{\beta}u(0)))''=0.\end{cases} \tag{2.9}$$

has a solution of the form

$$u(t) = \int_0^1 G(t,s)\phi_q(\int_0^1 \mathcal{H}(s,\tau)y(\tau)d\tau)ds, \tag{2.10}$$

where

$$\mathcal{H}(t,s) = \begin{cases} -\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{t}{1-\xi}(\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} - \frac{1}{\Gamma(\alpha)}(\xi-s)^{\alpha-1}), & s \le t,\ \xi \ge s \\ \frac{t}{1-\xi}(\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} - \frac{1}{\Gamma(\alpha)}(\xi-s)^{\alpha-1}), & t \le s,\ \xi \ge s \\ -\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{t}{1-\xi}(\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}, & s \le t,\ s \ge \xi \\ \frac{t}{1-\xi}(\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}, & t \le s,\ s \ge \xi, \end{cases} \tag{2.11}$$

and $G(t,s)$ is given by (2.3).

Proof. Applying integral I^α on the differential equation in (2.9) and using lemma (1.8), we obtain

$$\phi_p(D^\beta u(t)) = -I^\alpha y(t) + c_1 + c_2 t + c_3 t^2. \tag{2.12}$$

The boundary conditions $\phi_p(D^\beta u(0)) = 0, (\phi_p(D^\beta u(0)))'' = 0$ lead to $c_1 = 0 = c_3$ and the boundary condition $\phi_p(D^\beta u(\xi)) = (\phi_p(D^\beta u(1)))'$ yields $c_2 = \frac{1}{1-\xi}(I^{\alpha-1}y(1) - I^\alpha y(\xi))$. Consequently, (2.12) takes the form

$$\phi_p(D^\beta u(t)) = -I^\alpha y(t) + \frac{t}{1-\xi}(I^{\alpha-1}y(1) - I^\alpha y(\xi)), \tag{2.13}$$

which can be written as

$$\begin{aligned} \phi_p(D^\beta u(t)) = &-\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)ds \\ &+ \frac{t}{1-\xi}(\frac{1}{\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2}y(s)ds \\ &- \frac{1}{\Gamma(\alpha)}\int_0^\xi (\xi-s)^{\alpha-1}y(s)ds) \\ = &\int_0^1 \mathcal{H}(t,s)y(s)ds. \end{aligned} \tag{2.14}$$

The boundary value problem (2.9) reduces to the following problem

$$\begin{aligned} &D^\beta u(t) = \phi_q(\int_0^1 \mathcal{H}(t,s)y(s)ds) \\ &u(0) = 0, \quad \gamma u'(1) = u'(0), \quad u''(0) = 0 \end{aligned} \tag{2.15}$$

which in view of lemma (2.1) yields the required result

$$u(t) = \int_0^1 G(t,s)\phi_q(\int_0^1 \mathcal{H}(s,\tau)y(\tau)d\tau)ds.$$

□

Lemma 2.3. *The function $\mathcal{H}(t,s)$ defined by* (2.11) *is continuous on* $[0,1]\times[0,1]$ *and satisfies the following relations*

(A) $\mathcal{H}(t,s)\geq 0$, $\mathcal{H}(t,s)\leq \mathcal{H}(1,s)$, *for* $t,s\in[0,1]$
(B) $\mathcal{H}(t,s)\geq t^{\alpha-1}\mathcal{H}(1,s)$ *for* $t,s\in(0,1)$

Proof. Continuity of H clearly follows from the definition of H. For $0<s\leq t\leq\xi<1$, we have the following

$$\begin{aligned}
\mathcal{H}(t,s) &= -\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}+\frac{t}{1-\xi}(\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}-\frac{1}{\Gamma(\alpha)}(\xi-s)^{\alpha-1})\\
&= -t^{\alpha-1}\frac{(1-\frac{s}{t})^{\alpha-1}}{\Gamma(\alpha)}+\frac{t}{1-\xi}(\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}-\frac{1}{\Gamma\alpha}(\xi-s)^{\alpha-1})\\
&\geq -\frac{t^{\alpha-1}}{\Gamma\alpha}(1-s)^{\alpha-1}+\frac{t^{\alpha-1}}{1-\xi}(\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}-\frac{1}{\Gamma(\alpha)}(\xi-s)^{\alpha-1})\\
&= \frac{t^{\alpha-1}}{(1-\xi)\Gamma(\alpha)}(-(1-s)^{\alpha-1}(1-\xi)+(1-s)^{\alpha-2}(\alpha-1)-(\xi-s)^{\alpha-1})\\
&\geq \frac{t^{\alpha-1}}{(1-\xi)\Gamma(\alpha)}(-(1-s)^{\alpha-1}+\xi^{\alpha-1}(1-s)^{\alpha-1}-(\xi-s)^{\alpha-1}+\\
&(1-s)^{\alpha-2}(\alpha-1))\\
&\geq \frac{t^{\alpha-1}}{(1-\xi)\Gamma(\alpha)}(-(1-s)^{\alpha-1}+(\xi-s)^{\alpha-1}-(\xi-s)^{\alpha-1}+\\
&(1-s)^{\alpha-2}(\alpha-1))\geq 0.
\end{aligned}$$

The other cases can be deal similarly. Now,

$$\frac{\partial\mathcal{H}}{\partial t}(t,s)=\begin{cases}-\frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}+\frac{1}{1-\xi}(\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}-\frac{1}{\Gamma(\alpha)}(\xi-s)^{\alpha-1}), & s\leq t,\ \xi\geq s\\ \frac{1}{1-\xi}(\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}-\frac{1}{\Gamma(\alpha)}(\xi-s)^{\alpha-1}), & t\leq s,\ \xi\geq s\\ -\frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-)}+\frac{1}{1-\xi}(\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}, & s\leq t,\ s\geq\xi\\ \frac{1}{1-\xi}(\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}, & t\leq s,\ s\geq\xi,\end{cases}$$

Clearly, $\frac{\partial\mathcal{H}}{\partial t}(t,s)>0$ which implies that $\mathcal{H}(t,s)$ is an increasing function of t. Hence $\mathcal{H}(t,s)\leq\mathcal{H}(1,s)$.

Part (B) follows from the following

$$\begin{aligned}
\frac{\mathcal{H}(t,s)}{\mathcal{H}(1,s)} &= \frac{-\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}+\frac{t}{1-\xi}(\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}-\frac{1}{\Gamma(\alpha)}(\xi-s)^{\alpha-1})}{-\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}+\frac{1}{1-\xi}(\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}-\frac{1}{\Gamma(\alpha)}(\xi-s)^{\alpha-1})}\\
&\geq \frac{-t^{\alpha-1}\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}+\frac{t^{\alpha-1}}{1-\xi}(\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}-\frac{1}{\Gamma(\alpha)}(\xi-s)^{\alpha-1})}{-\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}+\frac{1}{1-\xi}(\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}-\frac{1}{\Gamma(\alpha)}(\xi-s)^{\alpha-1})}\\
&\geq t^{\alpha-1}\frac{-\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}+\frac{1}{1-\xi}(\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}-\frac{1}{\Gamma(\alpha)}(\xi-s)^{\alpha-1})}{-\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}+\frac{1}{1-\xi}(\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}-\frac{1}{\Gamma(\alpha)}(\xi-s)^{\alpha-1})}=t^{\alpha-1}.
\end{aligned}$$

□

Assume that the following hold:

(W1) $0 < \int_0^1 \mathcal{H}(1,\tau)a(\tau)d\tau < +\infty$.

(W2) There exist $0 < \delta < 1$ and $\rho > 0$ such that

$$f(x) \leq \delta L\phi_p(x), \text{ for } 0 \leq x \leq \rho, \tag{2.16}$$

where $0 < L \leq (\phi_p(\varpi_1)\delta \int_0^1 \mathcal{H}(1,s)a(s)ds)^{-1}$, $\varpi_1 = \frac{1}{\Gamma(\beta+1)} + \frac{\gamma}{(1-\gamma)\Gamma(\beta)}$.

(W3) There exist $b > 0$, such that

$$f(x) \leq M\phi_p(x), \text{ for } x > b,\ 0 < M < (\phi_p(\varpi_1 2^{q-1})\int_0^1 \mathcal{H}(1,\tau)a(\tau)d\tau)^{-1}. \tag{2.17}$$

(W4) $f(x)$ is non-decreasing in x.

(W5) There exist $0 \leq \theta < 1$ such that

$$f(kx) \geq (\phi_p(k))^\theta f(x), \text{ for any } 0 < k < 1 \text{ and } 0 < x < +\infty \tag{2.18}$$

2.1. **Existence and Uniqueness of solutions:**

Theorem 2.4. *Under the assumptions* $(W1)$ *and* $(W2)$*, the boundary value problem* (1.5) *has at least one positive solution.*

Proof. Define $K_1 = \{u \in C[0,1] : 0 \leq u(t) \leq \rho\}$ a closed convex set [9] and an operator $\mathcal{T} : K_1 \to C[0,1]$ by

$$\mathcal{T}u(t) = \int_0^1 G(t,s)\phi_q(\int_0^1 \mathcal{H}(s,\tau)a(\tau)f(u(\tau))d\tau)ds. \tag{2.19}$$

By lemma (2.2), u is a solution of the boundary value problem (1.5) if and only if u is a fixed point of $\mathcal{T}$. For any $u \in K_1$, using $(W2)$ and lemma (2.3), we obtain

$$\begin{aligned}
\mathcal{T}u(t) &= \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\phi_q(\int_0^1 \mathcal{H}(s,\tau)a(\tau)f(u(\tau))d\tau)ds \\
&+ \frac{t\gamma}{(1-\gamma)(\Gamma(\beta-1))}\int_0^1 (1-s)^{\beta-2}\phi_q(\int_0^1 \mathcal{H}(s,\tau)a(\tau)f(u(\tau))d\tau)ds \\
&\leq \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\phi_q(\int_0^1 \mathcal{H}(1,\tau)a(\tau)\delta L\phi_p(\rho)d\tau)ds \\
&+ \frac{t\gamma}{(1-\gamma)(\Gamma(\beta-1))}\int_0^1 (1-s)^{\beta-2}\phi_q(\int_0^1 \mathcal{H}(1,\tau)a(\tau)\delta L\phi_p(\rho))d\tau)ds \\
&\leq (\frac{1}{\Gamma(\beta+1)} + \frac{1}{(1-\gamma)}\frac{\gamma}{\Gamma(\beta)})\phi_q(\int_0^1 \mathcal{H}(1,\tau)a(\tau)d\tau)\phi_q(\delta)\phi_q(L)\rho \\
&= \varpi_1\phi_q(\int_0^1 \mathcal{H}(1,\tau)a(\tau)d\tau)\phi_q(\delta)\phi_q(L)\rho \leq \rho,
\end{aligned}$$

which implies that $\mathcal{T}(K_1) \subseteq K_1$ and also demonstrate that $\mathcal{T}$ is uniformly bounded. In order to show the compactness of the operator $\mathcal{T}$, we only need to show that it is equicontinuous. For $u \in K_1$ and t_1, $t_2 \in [0,1]$ with $t_1 \leq t_2$, we have

$$\mid \mathcal{T}u(t_2) - \mathcal{T}u(t_1) \mid = \mid \int_0^1 (G(t_2,s) - G(t_1,s))\phi_q(\int_0^1 \mathcal{H}(s,\tau)a(\tau)f(u(\tau))d\tau)ds \mid$$
$$\leq \int_0^1 \mid G(t_2,s) - G(t_1,s) \mid \phi_q(\int_0^1 \mathcal{H}(s,\tau)a(\tau)f(u(\tau))d\tau)ds,$$

which in view of (2.8) implies that $\mid \mathcal{T}u(t_2) - \mathcal{T}u(t_1) \mid \to 0$ as $t_1 \to t_2$. Hence by Arzela Ascoli theorem, $\mathcal{T}$ is comapact. By Schauder fixed point theorem $\mathcal{T}$ has a fixed point in K_1. □

Theorem 2.5. *Under the assumption* $(W1)$ *and* $(W3)$*, the boundary value problem for fractional differential equation* (1.5) *has at least one positive solution.*

Proof. Let $b > 0$ as given in $(W3)$. Define $\chi = max_{0 \leq x \leq b} f(x)$. Then $f(x) \leq \chi$ for $0 \leq x \leq b$. In view of $(W3)$, we have

$$\varpi_1 2^{q-1}\phi_q(M)\phi_q(\int_0^1 \mathcal{H}(1,\tau)a(\tau)d\tau) < 1.$$

Choose $b^* > b$ large enough such that

$$\varpi_1 2^{q-1}(\phi_q(\chi) + \phi_q(M)b^*)\phi_q(\int_0^1 \mathcal{H}(1,\tau)a(\tau))d\tau) < b^*. \tag{2.20}$$

Define $K_1 = \{u \in C[0,1] : 0 \leq u(t) \leq b^* \text{ on } [0,1]\}$. For $u \in K_1$, define $S_1 = \{t \in [0,1] : 0 \leq u(t) \leq b\}$, $S_2 = \{t \in [0,1] : b < u(t) \leq b^*\}$. Then we have $S_1 \cup S_2 = [0,1]$ and $S_1 \cap S_2 = \emptyset$ and in view of (2.17), it follows that

$$f(u(t)) \leq M\phi_p(u(t)) \leq M\phi_p(b^*) \text{ for } t \in S_2. \tag{2.21}$$

For $u \in K_1$, using Lemma (2.3) and (2.17), it follows that

$$\begin{aligned}
\mathcal{T}u(t) &= \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\phi_q(\int_0^1 \mathcal{H}(s,\tau)a(\tau)f(u(\tau))d\tau)ds \\
&+ \frac{t\gamma}{1-\gamma}\frac{1}{\Gamma(\beta-1)}\int_0^1 (1-s)^{\beta-2}\phi_q(\int_0^1 \mathcal{H}(s,\tau)a(\tau)f(u(\tau))d\tau)ds \\
&\leq \varpi_1\phi_q(\int_0^1 \mathcal{H}(1,\tau)a(\tau)f(u(\tau))d\tau) \\
&= \varpi_1\phi_q(\int_{S_1} \mathcal{H}(1,\tau)a(\tau)f(u(\tau))d\tau + \int_{S_2} \mathcal{H}(1,\tau)a(\tau)f(u(\tau))d\tau) \\
&\leq \varpi_1\phi_q(\chi\int_{S_1} \mathcal{H}(1,\tau)a(\tau))d\tau + M\phi_p(b^*)\int_{S_2} \mathcal{H}(1,\tau)a(\tau))d\tau) \\
&\leq \varpi_1\phi_q(\chi + M\phi_p(b^*))\phi_q(\int_0^1 \mathcal{H}(1,\tau)a(\tau))d\tau).
\end{aligned}$$

From (2.20) and by the help of inequality $(a+b)^r \leq 2^r(a^r + b^r)$ for any $a, b, r > 0$, we have

$$0 \leq \mathcal{T}u(t) \leq \varpi_1 2^{q-1}(\phi_q(\chi) + \phi_q(M)b^*)\phi_q(\int_0^1 \mathcal{H}(1,\tau)a(\tau))d\tau) \leq b^*,$$

which implies that $\mathcal{T}(K_2) \subseteq K_2$. Hence by Schauder fixed point theorem $\mathcal{T}$ has a fixed point $u \in K_1$. $\square$

Theorem 2.6. *Assume that* $(W1)$, $(W4)$ *and* $(W5)$ *hold. Then the fractional differential equation* (1.5) *has a unique positive solution*

Proof. Define $P = \{u \in C[0,1] : u(t) \geq 0 \text{ on } [0,1]\}$. Then P is a normal solid cone in $C[0,1]$ with $P^o = \{u \in C[0,1] : u(t) > 0 \text{ on } [0,1]\}$. Let $\mathcal{T} : P \to C[0,1]$ be defined by (2.19) We show that $\mathcal{T}$ is $\theta-$concave increasing operator. For $u_1,\ u_2 \in P$ with $u_1 \geq u_2$ we have $\mathcal{T}(u_1) \geq \mathcal{T}(u_2)$ on $[0,1]$, and form $f(ku) \geq \phi_q(k^\theta) f(u)$ we have the following estimates

$$\begin{aligned}\mathcal{T}(ku(t)) &\geq \int_0^1 G(t,s)\phi_q(\int_0^1 H(t,\tau)\phi_q(k^\theta)f(u)d\tau)ds\\ &= k^\theta \int_0^1 G(t,s)\phi_q(\int_0^1 H(t,\tau)f(u)d\tau)ds = k^\theta \mathcal{T}(u(t)),\end{aligned}$$

which implies that $\mathcal{T}$ is $\theta-$concave operator. Thus $\mathcal{T}$ has a unique fixed point $\square$

Example 2.7. Consider the following boundary value problem

$$\begin{aligned}&D^{2.5}(\phi_p(D^{2.5}u(t))) + tu(t) = 0,\\ &u(0) = 0,\ 1/2u'(1) = u'(0),\ u''(0) = 0\\ &\phi_p(D^{2.5}u(0)) = 0,\ \phi_p(D^{2.5}u(1/2)) = (\phi_p(D^{2.5}u(1)))',\ (\phi_p(D^{2.5}u(0)))'' = 0.\end{aligned} \tag{2.22}$$

Here we have $\alpha = \beta = 2.5$, $\xi = \gamma = 1/2$, $a(t) = t$, $f(u(t)) = u(t)$. By simple computation, we obtain $0 < L \leq 1.7807$, $\delta = 1/2$. Choose $L = 1$ and $\delta = 1/2$, the conditions $(W1)$ and $(W2)$ are satisfied. Hence, by theorem (2.4), the fractional differential equation (2.22) has at least one positive solution.

Example 2.8. For the following boundary value problem

$$\begin{aligned}&D^{2.5}(\phi_p(D^{2.5}u(t))) + t\sqrt[3]{u(t)} = 0,\\ &u(0) = 0,\ \gamma u'(1) = u'(0),\ u''(0) = 0\\ &\phi_p(D^{2.5}u(0)) = 0,\ \phi_p(D^{2.5}u(1/2)) = (\phi_p(D^{2.5}u(1)))',\\ &(\phi_p(D^{2.5}u(0)))'' = 0,\end{aligned} \tag{2.23}$$

we have $\alpha = \beta = 2.5$, $\xi = \eta = 1/2$, $a(t) = t$, $f(u(t)) = \sqrt[3]{u}(t)$ and by simple computation we get $M < .4337$ and thus by choosing $M = .3333$, $b = 1$ and $q = 2$, we see that (2.23) satisfy $(W1)$ and $(W3)$. Hence by theorem (2.4), the fractional differential equation (2.23) has at least one positive solution.

Example 2.9.

$$\begin{aligned}&D^{2.5}(\phi_p(D^{2.5}u(t))) + t\sqrt{u(t)} = 0,\\ &u(0) = 0,\ \gamma u'(1) = u'(0),\ u''(0) = 0\\ &\phi_p(D^{2.5}u(0)) = 0,\ \phi_p(D^{2.5}u(1/2)) = (\phi_p(D^{2.5}u(1)))',\ (\phi_p(D^{2.5}u(0)))'' = 0.\end{aligned} \tag{2.24}$$

For the uniqueness of solution for fractional differential equation (2.24), we apply theorem (2.6). In equation (2.24), we have $\alpha = \beta = 2.5$, $\xi = \gamma = 1/2$, $a(t) = t$, $f(u(t)) = \sqrt{u(t)}$ it is clear that (2.24) satisfy conditions $(W1)$, $(W4)$. Also considering $\theta = 1/2$, $W5$ is satisfied. Thus by theorem (2.6), fractional differential equation (2.24) has a unique solution.

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