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Numerical solution of delay differential equations via operational matrices of hybrid of block-pulse functions and Bernstein polynomials

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Abstract In this paper, we introduce hybrid of block-pulse functions and Bernstein polynomials and derive operational matrices of integration, dual, differentiation, product and delay of these hybrid functions by a general procedure that can be used for other polynomials or orthogonal functions. Then, we utilize them to solve delay differential equations and time-delay system. The method is based upon expanding various time-varying functions as their truncated hybrid functions. Illustrative examples are included to demonstrate the validity, efficiency and applicability of the method.

Keywords. Delay differential equation, Bernstein polynomial, Hybrid of block-pulse function, Operational matrix.

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1. Introduction

Delays occur frequently in biological, chemical, electronic and transportation systems [10]. Time-delay systems are therefore a very important class of systems whose control and optimization have been of interest to many investigators. Orthogonal functions and polynomial series have received considerable attention in dealing with various problems of dynamic systems. The main characteristic of this technique is that it reduces these problems to those of solving a system of algebraic equations, thus greatly simplifying the problems [2, 3, 7, 9, 12, 19]. The approach is based on converting the underlying differential equations into integral equations through integration, approximating various signals involved in the equation by truncated polynomials series and using the operational matrices to eliminate the integral, derivation and delay operations. Special attention has recently been given to applications of Walsh function [3], Haar wavelets [6], Chebyshev polynomials [7], Bessel functions [16], Legendre polynomials [2] , Legendre wavelets [17] , Sine-cosine wavelets [18] and Bernstein polynomials [23]. In recent years the different kinds of hybrid functions [4, 5, 8, 9, 11, 12, 13, 14, 19, 20] were applied to solve the delay problem.

Bernstein polynomials have some properties that distinguish it from other polynomials for example continuity and partition of unity property. All of the Bernstein polynomials vanish at the initial and end points of the interval $[a, b]$ except for the first polynomial and the last polynomial are equal to 1 at $x = a$ and $x = b$, respectively, which provides greater flexibility to impose boundary conditions. The Bernstein polynomials, although not based on orthogonal polynomials, can also be applied to analyze various problems. Specifically, in [24, 25] Bernstein polynomials have been used for solving the partial differential equation.

In this paper, hybrid of block-pulse functions and Bernstein polynomials are introduced and derived operational matrices of integration $\overline{P}$, dual $\overline{Q}$, differentiation $\overline{D}$, product $\widehat{C}$ and delay Del by a general procedure. These matrices are applied to evaluate the solution of delay differential equation or delay differential system by expanding the candidate function as hybrid functions with unknown coefficients and reducing the delay problem to a set of algebraic equations.

This paper is organized as follows: In section 2 we describe the basic formulation of Bernstein polynomials which is required for our subsequent development. Section 3 is devoted to the definition of hybrid of block-pulse functions and Bernstein polynomials and function approximation using these functions. In section 4 we calculate the operational matrices of integration, dual, differentiation, product and delay. In section 5, we report our numerical findings which demonstrate the validity, accuracy, efficiency and applicability of the operational matrices by considering some test problems. Section 6 consists of a brief summary.

2. Properties of Bernstein polynomials

The Bernstein polynomials of mth-degree are defined on the interval $[a, b]$ as [1]

$$B_{i,m}(x) = \binom{m}{i} \frac{(x-a)^i (b-x)^{m-i}}{(b-a)^m}, \quad i = 0, 1, \ldots, m,$$

where

$$\binom{m}{i} = \frac{m!}{i!(m-i)!}.$$

These Bernstein polynomials form a basis on $L^2[a,b]$. There are $m+1$ polynomials of mth-degree. For convenience, we set $B_{i,m}(x)=0$, if $i<0$ or $i>m$. A recursive definition can also be used to generate the Bernstein polynomials over $[a,b]$ so that the ith Bernstein polynomial of mth-degree can be written

$$B_{i,m}(x)=\frac{(b-x)}{b-a}B_{i,m-1}(x)+\frac{(x-a)}{b-a}B_{i-1,m-1}(x).$$

It can readily be shown that each of the Bernstein polynomials is positive and the sum of all the Bernstein polynomials is unity for all real $x\in[a,b]$, i.e., $\sum_{i=0}^{m}B_{i,m}(x)=1$ (unity partition property). It is easy to show that any given polynomial of mth-degree can be expanded in terms of these basis functions.

3. Properties of hybrid functions

3.1. Hybrid of block-pulse functions and Bernstein polynomials. Since interval $[a,b)$ can be shifted to $[0,1)$, therefor we concentrate on $[0,1)$. Hybrid of block-pulse functions and Bernstein polynomials are defined on $[0,1)$ for $i=0,1,\ldots,m$ and $n=0,1,\ldots,N-1$,

$$\psi_{i,n}(t)=\begin{cases} B_{i,m}(kt-n) & \frac{n}{k}\leq t<\frac{n+1}{k}\\ 0 & \text{otherwise,}\end{cases} \tag{3.1}$$

where m is the degree of Bernstein polynomial on $[0,1]$, n is transmission parameter, N denotes the number of subinterval of $[0,1]$ and the parameters k, N will be specified.

3.2. Function Approximation. Suppose that $H=L^2[0,1]$ and $\{\psi_{i,n}(t)\}_{i=0,n=0}^{m,N-1}\subset H$ be the set of hybrid functions of Bernstein polynomials of mth-degree and

$$Y=Span\{\psi_{i,n}(t)\mid i=0,1,\cdots m,\quad n=0,1,\cdots,N-1\}$$

and f be an arbitrary element in H. Since Y is a finite dimensional vector space, f has the unique best approximation out of Y such as $y_0\in Y$, that is

$$\exists y_0\in Y;\ \forall y\in Y\ ||f-y_0||_2\leq||f-y||_2,$$

where $||f||_2^2=<f,f>=\int_0^1 f^2(x)dx$.

Since $y_0\in Y$, there exist the unique coefficients $c_{i,n}$ such that

$$f\simeq y_0=\sum_{n=0}^{N-1}\sum_{i=0}^{m}c_{i,n}\psi_{i,n}=c^T\phi,$$

where

$$\phi^T=[\psi_{0,0}\ ,\psi_{1,0}\ ,\cdots,\psi_{m-1,0}\ ,\psi_{m,0},\cdots,\psi_{0,N-1},\ \psi_{1,N-1}\ ,\cdots,\psi_{m-1,N-1}\ ,\psi_{m,N-1}],$$

$$c^T=[c_{0,0}\ ,c_{1,0}\ ,\cdots,c_{m-1,0}\ ,c_{m,0}\ ,\cdots,c_{0,N-1}\ ,c_{1,N-1}\ ,\cdots,c_{m-1,N-1}\ ,c_{m,N-1}],$$

and c^T can be obtained by

$$c^T < \phi\ ,\ \phi >=< f,\ \phi >,$$

where

$$< f,\ \phi >= \int_0^1 f(x)\phi(x)^T dx$$

and $< \phi\ ,\ \phi >$ is a $N(m+1) \times N(m+1)$ matrix which is said dual operational matrix of ϕ and denoted by $\overline{Q}$

$$\overline{Q} =< \phi\ ,\ \phi >= \int_0^1 \phi(x)\phi(x)^T dx,$$

then

$$c^T = \left(\int_0^1 f(x)\phi(x)^T dx \right) (\ \overline{Q}\ )^{-1}. \tag{3.2}$$

In the following lemma we present an upper bound for the error approximation.

Lemma 3.1. *Suppose that the function $g : [0, 1) \to \mathbb{R}$ is $m+1$ times continuously differentiable, $g \in C^{m+1}[t_0, t_f]$ and $Y = Span\{\psi_{i,n}(t) \mid i = 0, 1, \cdots m, \quad n = 0, 1, \cdots, N-1\}$. If $c^T\phi$ is the best approximation g out of Y then the mean error bounded is presented as follows:*

$$||g - c^T\phi||_2 \leq \frac{M}{(m+1)!\ k^{m+1}\ \sqrt{2m+3}}\ ,$$

where $M = \max_{x \in [t_0, t_f]} |g^{(m+1)}(x)|$.

Proof. We consider the Taylor polynomial of order m for function g on $[\frac{n}{k}, \frac{n+1}{k})$

$$y_n(x) = g(\frac{n}{k}) + g'(\frac{n}{k})(x - \frac{n}{k}) + \cdots + g^{(m)}(\frac{n}{k})\frac{(x - \frac{n}{k})^m}{m!}$$

for $n = 0, 1, \cdots, N-1$ which we know

$$|g(x) - y_n(x)| \leq |g^{(m+1)}(\eta)| \frac{(x - \frac{n}{k})^{m+1}}{(m+1)!} \tag{3.3}$$

where $\eta \in (\frac{n}{k}, \frac{n+1}{k})$. Since $c^T\phi$ is the best approximation g out of Y, $y_n \in Y$ and using (3.3) we have

$$||g - c^T\phi||_2^2 = \int_0^1 |g(x) - c^T\phi(x)|^2 dx = \sum_{n=0}^{N-1} \int_{\frac{n}{k}}^{\frac{n+1}{k}} |g(x) - c^T\phi(x)|^2 dx$$

$$\leq \sum_{n=0}^{N-1} \int_{\frac{n}{k}}^{\frac{n+1}{k}} |g(x) - y_n(x)|^2 dx \leq \sum_{n=0}^{N-1} \int_{\frac{n}{k}}^{\frac{n+1}{k}} \left[g^{(m+1)}(\eta) \frac{(x - \frac{n}{k})^{m+1}}{(m+1)!} \right]^2 dx$$

$$\leq \frac{M^2}{(m+1)!\ ^2} \sum_{n=0}^{N-1} \int_{\frac{n}{k}}^{\frac{n+1}{k}} (x - \frac{n}{k})^{2m+2} dx = \frac{M^2}{[(m+1)!\ ]^2\ k^{2m+2}\ (2m+3)},$$

and by taking square root we have the above bound. □

The presented upper bound of the error depends on $\frac{1}{(m+1)!\ k^{m+1}\ \sqrt{2m+3}}$ which shows that the error reduces to zero very fast as m increase. This is one of the advantages of hybrid of block-pulse functions and Bernstein polynomials.

4. Operational matrices of hybrid of block-pulse functions and Bernstein polynomials

4.1. **Operational matrix of integration.** The operational matrix of integration $\overline{P}$ is given by

$$\int_0^x \phi(t)dt \simeq \overline{P}\phi(x), \quad 0 \leq x < 1.$$

For obtaining an explicit formula for $\overline{P}$ we denote the vector

$$B(kx-n) = \begin{bmatrix} B_{0,m}(kx-n) \\ B_{1,m}(kx-n) \\ \vdots \\ B_{m,m}(kx-n) \end{bmatrix} = \begin{bmatrix} \psi_{0,n}(x) \\ \psi_{1,n}(x) \\ \vdots \\ \psi_{m,n}(x) \end{bmatrix}$$

for $n = 0, 1, \cdots, N-1$

$$\phi(x) = \begin{cases} B(kx) & 0 \leq x < \frac{1}{k} \\ B(kx-1) & \frac{1}{k} \leq x < \frac{2}{k} \\ B(kx-2) & \frac{2}{k} \leq x < \frac{3}{k} \\ \vdots & \\ B(kx-(N-1)) & \frac{N-1}{k} \leq x < 1\ . \end{cases}, \tag{4.1}$$

It is easy to see that:

$$\int_0^1 (1-x)^r x^i dx = \frac{1}{(r+i+1)\binom{r+i}{i}}, \qquad i, r \in \mathbb{N} \cup \{0\}$$

therefore

$$\int_0^1 B_{i,m}(x)dx = \frac{1}{m+1}, \qquad i = 0, 1, \cdots, m$$

so

$$\int_0^1 B(x)dx = \begin{bmatrix} \frac{1}{m+1} \\ \frac{1}{m+1} \\ \vdots \\ \frac{1}{m+1} \end{bmatrix}$$

then

$$\int_0^1 B(kx-n)dx = \begin{bmatrix} \frac{1}{k(m+1)} \\ \frac{1}{k(m+1)} \\ \vdots \\ \frac{1}{k(m+1)} \end{bmatrix}, \quad n = 0, 1, \cdots, N-1. \tag{4.2}$$

On the other hand we know

$$\int_0^x B(t)dt \simeq PB(x), \quad 0 \leq x \leq 1 \tag{4.3}$$

which P is the operational matrix of integration of $B(x)$ and the details of obtaining this matrix is given in [23]. Now, we want to obtain the operational matrix of integration $\overline{P}$ using (4.2), (4.3) and the property of partition of unity of Bernstein polynomial

$$\int_0^x B(kt)dt = \begin{cases} \int_0^x B(kt)dt & 0 \leq x < \frac{1}{k} \\ \int_0^{\frac{1}{k}} B(kt)dt + \int_{\frac{1}{k}}^x B(kt)dt & \frac{1}{k} \leq x < \frac{2}{k} \\ \vdots & \\ \int_0^{\frac{1}{k}} B(kt)dt + \int_{\frac{1}{k}}^{\frac{2}{k}} B(kt)dt + \cdots + \int_{\frac{N-1}{k}}^x B(kt)dt & \frac{N-1}{k} \leq x < \frac{N}{k} \end{cases}$$

$$= \begin{cases} \frac{P}{k} B(kx), & 0 \leq x < \frac{1}{k} \\ \begin{bmatrix} \frac{1}{k(m+1)} \\ \frac{1}{k(m+1)} \\ \vdots \\ \frac{1}{k(m+1)} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \frac{\overline{1}}{k(m+1)} B(kx-1), & \frac{1}{k} \leq x < \frac{2}{k} \\ \vdots & \\ \begin{bmatrix} \frac{1}{k(m+1)} \\ \frac{1}{k(m+1)} \\ \vdots \\ \frac{1}{k(m+1)} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \ldots + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \frac{\overline{1}}{k(m+1)} B(kx-(N-1)), & \frac{N-1}{k} \leq x < \frac{N}{k} \end{cases}$$

which $\overline{1}$ is a matrix $(m+1)\times(m+1)$ that all of its elements is 1, therefore

$$\int_0^x B(kt)dt = \left[\frac{P}{k}, \frac{\overline{1}}{k(m+1)}, \cdots, \frac{\overline{1}}{k(m+1)}\right] \begin{bmatrix} B(kx) \\ B(kx-1) \\ \vdots \\ B(kx-(N-1)) \end{bmatrix}$$

$$= \left[\frac{P}{k}, \frac{\overline{1}}{k(m+1)}, \cdots, \frac{\overline{1}}{k(m+1)}\right] \phi(x).$$

Similarly, we have

$$\int_0^x B(kt-1)dt = \begin{cases} \int_0^x B(kt-1)dt & 0 \le x < \frac{1}{k} \\ \int_0^{\frac{1}{k}} B(kt-1)dt + \int_{\frac{1}{k}}^{x} B(kt-1)dt & \frac{1}{k} \le x < \frac{2}{k} \\ \vdots & \\ \int_0^{\frac{1}{k}} B(kt-1)dt + \int_{\frac{1}{k}}^{\frac{2}{k}} B(kt-1)dt + \cdots + \int_{\frac{N-1}{k}}^{x} B(kt-1)dt & \frac{N-1}{k} \le x < \frac{N}{k} \end{cases}$$

$$= \begin{cases} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \overline{0}\, B(kx) & 0 \le x < \frac{1}{k} \\ \frac{P}{k} B(kx-1) & \frac{1}{k} \le x < \frac{2}{k} \\ \begin{bmatrix} \frac{1}{k(m+1)} \\ \frac{1}{k(m+1)} \\ \vdots \\ \frac{1}{k(m+1)} \end{bmatrix} = \frac{\overline{1}}{k(m+1)} B(kx-2) & \frac{2}{k} \le x < \frac{3}{k} \\ \vdots & \\ \begin{bmatrix} \frac{1}{k(m+1)} \\ \frac{1}{k(m+1)} \\ \vdots \\ \frac{1}{k(m+1)} \end{bmatrix} = \frac{\overline{1}}{k(m+1)} B(kx-(N-1)) & \frac{N-1}{k} \le x < \frac{N}{k} \end{cases}$$

then

$$\int_0^x B(kt-1)dt = \left[\overline{0}, \frac{P}{k}, \frac{\overline{1}}{k(m+1)}, \cdots, \frac{\overline{1}}{k(m+1)}\right] \phi(x)$$

which $\overline{0}$ is the zero matrix $(m+1)\times(m+1)$. The same process can be done for the rest of the vectors, therefore the operational matrix of integration $\overline{P}$ is obtained as follows

$$\overline{P} = \begin{bmatrix} \frac{P}{k} & \frac{\overline{1}}{k(m+1)} & \frac{\overline{1}}{k(m+1)} & \cdots & \frac{\overline{1}}{k(m+1)} \\ \overline{0} & \frac{P}{k} & \frac{\overline{1}}{k(m+1)} & \cdots & \frac{\overline{1}}{k(m+1)} \\ \overline{0} & \overline{0} & \frac{P}{k} & \cdots & \frac{\overline{1}}{k(m+1)} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \overline{0} & \overline{0} & \overline{0} & \cdots & \frac{P}{k} \end{bmatrix}.$$

4.2. **Dual operational matrix.** We define the dual operational matrix of ϕ in the preceding section that

$$\overline{Q} = \int_0^1 \phi(x)\phi^T(x)dx.$$

In [23] the dual operational matrix of $B(x)$ is presented

$$Q = \int_0^1 B(x)B^T(x)dx.$$

We have

$$\int_0^1 B(kx-n)B(kx-n)^T dx = \int_{\frac{n}{k}}^{\frac{n+1}{k}} B(kx-n)B(kx-n)^T dx = \frac{Q}{k}, \tag{4.4}$$

$$\int_0^1 B(kx-n)B(kx-j)^T dx = \overline{0}\ , \quad n \neq j \tag{4.5}$$

for $j, n = 0, 1, \cdots, N-1$. By using (4.4) and (4.5)

$$\overline{Q} = \int_0^1 \phi(x)\phi^T(x)dx$$

$$= \begin{bmatrix} \int_0^1 B(kx)B(kx)^T dx & \cdots & \int_0^1 B(kx)B(kx-(N-1))^T dx \\ \int_0^1 B(kx-1)B(kx)^T dx & \cdots & \int_0^1 B(kx-1)B(kx-(N-1))^T dx \\ \vdots & \cdots & \vdots \\ \int_0^1 B(kx-(N-1))B(kx)^T dx & \cdots & \int_0^1 B(kx-(N-1))B(kx-(N-1))^T dx \end{bmatrix}$$

$$= \frac{1}{k}\begin{bmatrix} Q & \overline{0} & \cdots & \overline{0} \\ \overline{0} & Q & \cdots & \overline{0} \\ \vdots & \vdots & \cdots & \vdots \\ \overline{0} & \overline{0} & \cdots & Q \end{bmatrix}$$

4.3. **Operational matrix of differentiation.** The operational matrix of differentiation $\overline{D}$ is given by

$$\frac{d\phi(x)}{dx} = \overline{D}\phi(x).$$

We have

$$\frac{dB(x)}{dx} = D\ B(x) \tag{4.6}$$

which D is the operational matrix of differentiation of $B(x)$ and the details of obtaining this matrix is given in [23].

The operational matrix of differentiation $\overline{D}$ is obtained using (4.6) as follows

$$\frac{d}{dx}\phi(x) = \begin{cases} \frac{d}{dx}B(kx) & 0 \le x < \frac{1}{k} \\ \frac{d}{dx}B(kx-1) & \frac{1}{k} \le x < \frac{2}{k} \\ \frac{d}{dx}B(kx-2) & \frac{2}{k} \le x < \frac{3}{k} \\ \vdots & \\ \frac{d}{dx}B(kx-(N-1)) & \frac{N-1}{k} \le x < \frac{N}{k} \end{cases}$$

$$= \begin{cases} kD\ B(kx) & 0 \le x < \frac{1}{k} \\ kD\ B(kx-1) & \frac{1}{k} \le x < \frac{2}{k} \\ kD\ B(kx-2) & \frac{2}{k} \le x < \frac{3}{k} \\ \vdots & \\ kD\ B(kx-(N-1)) & \frac{N-1}{k} \le x < \frac{N}{k} \end{cases}$$

so

$$\overline{D} = k\begin{bmatrix} D & \overline{0} & \overline{0} & \cdots & \overline{0} \\ \overline{0} & D & \overline{0} & \cdots & \overline{0} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \overline{0} & \overline{0} & \overline{0} & \cdots & D \end{bmatrix}.$$

4.4. **Operational matrix of product.** Suppose that $C^T = [C_0^T, C_1^T, \ldots, C_{N-1}^T]$ is an arbitrary $1 \times N(m+1)$ matrix which C_i^T is $1 \times (m+1)$ matrix for $i = 0, 1, \ldots, N-1$, then $\widehat{C}$ is $N(m+1) \times N(m+1)$ operational matrix of product whenever

$$C^T\phi(x)\phi(x)^T \simeq \phi(x)^T\widehat{C}.$$

We know

$$C_i^T B(x)B(x)^T \simeq B(x)^T\widehat{C_i}, \quad i = 0, 1, \ldots, N-1$$

which $\widehat{C_i}$ is operational matrix of product of Bernstein polynomials presented in [23], then

$$C^T\phi(x)\phi^T(x) =$$

$$C^T \begin{bmatrix} B(kx)B(kx)^T & \overline{0} & \overline{0} & \cdots & \overline{0} \\ \overline{0} & B(kx-1)B(kx-1)^T & \overline{0} & \cdots & \overline{0} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \overline{0} & \overline{0} & \overline{0} & \cdots & B(kx-(N-1))B(kx-(N-1))^T \end{bmatrix} =$$

$$\begin{bmatrix} C_0^T B(kx)B(kx)^T & \overline{0} & \overline{0} & \cdots & \overline{0} \\ \overline{0} & C_1^T B(kx-1)B(kx-1)^T & \overline{0} & \cdots & \overline{0} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \overline{0} & \overline{0} & \overline{0} & \cdots & C_{N-1}^T B(kx-(N-1))B(kx-(N-1))^T \end{bmatrix}$$

$$\simeq \begin{bmatrix} B(kx)^T\widehat{C_0} & \overline{0} & \overline{0} & \cdots & \overline{0} \\ \overline{0} & B(kx-1)^T\widehat{C_1} & \overline{0} & \cdots & \overline{0} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \overline{0} & \overline{0} & \overline{0} & \cdots & B(kx-(N-1))^T\widehat{C_{N-1}} \end{bmatrix} = \phi(x)^T\widehat{C}$$

which

$$\widehat{C} = \begin{bmatrix} \widehat{C_0} & \overline{0} & \overline{0} & \cdots & \overline{0} \\ \overline{0} & \widehat{C_1} & \overline{0} & \cdots & \overline{0} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \overline{0} & \overline{0} & \overline{0} & \cdots & \widehat{C_{N-1}} \end{bmatrix},$$

which $\overline{0}$ is $(m+1)\times(m+1)$ zero matrix.

4.5. **Operational matrix of delay.** The operational matrix of delay Del is given by

$$\phi(x-\tau) \simeq (Del)\phi(x), \quad \tau > 0\ .$$

Suppose that τ be a positive real number, then we let [15]

$$k = \frac{1}{\tau}\,, \quad N = \begin{cases} \frac{1}{\tau} & \frac{1}{\tau} \in \mathbb{Z} \\ 1 + [\frac{1}{\tau}] & \text{otherwise,} \end{cases} \tag{4.7}$$

which $[.]$ denotes greatest integer smaller than or equal to a number. From (4.1)

$$\phi(x-\tau) = \begin{cases} B(k(x-\tau)) & 0 \le x-\tau < \frac{1}{k} \\ B(k(x-\tau)-1) & \frac{1}{k} \le x-\tau < \frac{2}{k} \\ B(k(x-\tau)-2) & \frac{2}{k} \le x-\tau < \frac{3}{k} \\ \vdots & \vdots \\ B(k(x-\tau)-(N-1)) & \frac{N-1}{k} \le x-\tau < \frac{N}{k} \end{cases}$$

$$= \begin{cases} B(kx-1) & \frac{1}{k} \leq x < \frac{2}{k} \\ B(kx-2) & \frac{2}{k} \leq x < \frac{3}{k} \\ B(kx-3) & \frac{3}{k} \leq x < \frac{4}{k} \\ \vdots & \vdots \\ B(kx-(N-1)) & \frac{N-1}{k} \leq x < \frac{N}{k} \\ B(kx-N) & \frac{N}{k} \leq x < \frac{N+1}{k} \end{cases}$$

thus

$$Del = \begin{bmatrix} \overline{0} & I & \overline{0} & \overline{0} & \cdots & \overline{0} \\ \overline{0} & \overline{0} & I & \overline{0} & \cdots & \overline{0} \\ \overline{0} & \overline{0} & \overline{0} & I & \cdots & \overline{0} \\ \vdots & & \vdots & \vdots & \ddots & \vdots \\ \overline{0} & \overline{0} & \overline{0} & \overline{0} & \cdots & I \\ \overline{0} & \overline{0} & \overline{0} & \overline{0} & \cdots & \overline{0} \end{bmatrix}$$

which I and $\overline{0}$ are $(m+1)\times(m+1)$ identity matrix and $(m+1)\times(m+1)$ zero matrix, respectively.

5. Illustrative example

In this section some examples are given to demonstrate the applicability and accuracy of our method. In all examples the package of Mathematica (7.0) has been used to solve the test problems considered in this paper.

Example 1. Consider the delay differential equation

$$\begin{aligned} \dot{x}(t) &= 1 + x(t-\tfrac{\sqrt{2}}{2})\ , \qquad && 0 \leq t < 1 \\ x(t) &= 0\ , && -\tfrac{\sqrt{2}}{2} \leq t < 0\ , \\ x(0) &= 1\ . \end{aligned} \tag{5.1}$$

with the exact solution

$$x(t) = \begin{cases} 1+t, & 0 \leq t < \frac{\sqrt{2}}{2} \\ \frac{5}{4} + \frac{\sqrt{2}}{2} - \sqrt{2} + (2-\frac{\sqrt{2}}{2})t + \frac{t^2}{2}, & \frac{\sqrt{2}}{2} \leq t < \sqrt{2}\,. \end{cases}$$

Since $\tau = \frac{\sqrt{2}}{2}$ and using (4.7), we let $k = \sqrt{2}$ and $N = 2$ in (4.1). We approximate the unknown function $\dot{x}(t)$ as

$$\dot{x}(t) = c^T\phi(t) \tag{5.2}$$

which c is the unknown vector, so

$$x(t) = c^T\overline{P}\phi(t) + 1 = (c^T\overline{P} + d^T)\phi(t)$$

where $1 = d^T\phi(t)$. Using the delay operational matrix

$$x(t - \frac{\sqrt{2}}{2}) = (c^T\overline{P} + d^T)\phi(t - \frac{\sqrt{2}}{2}) = (c^T\overline{P} + d^T)(Del)\phi(t). \qquad (5.3)$$

Substituting (5.2) and (5.3) in (5.1) we have

$$c^T = d^T + (c^T\overline{P} + d^T)(Del)$$

then

$$c^T = (d^T + d^T.Del)\ (I - \overline{P}(Del))^{-1}, \qquad (5.4)$$

which I is identity matrix with appropriate dimension. The equation set (14) is solved by $m = 2$ and $m = 3$ and the error values in some points is presented in Table 1 and the the error function is plotted in Figure 1 for $m = 3$. Numerical findings show the high accuracy of aproximate solution despite $\tau = \frac{\sqrt{2}}{2}$.

t	$m = 2$	$m = 3$
0	-1.01782$\times 10^{-10}$	8.88178$\times 10^{-16}$
0.1	-1.86387$\times 10^{-11}$	2.22045$\times 10^{-16}$
0.2	8.32948$\times 10^{-11}$	4.44089$\times 10^{-16}$
0.3	2.04019$\times 10^{-10}$	2.22045 $\times 10^{-16}$
0.4	3.43533$\times 10^{-10}$	4.44089 $\times 10^{-16}$
0.5	5.01838$\times 10^{-10}$	4.44089$\times 10^{-16}$
0.6	6.78933$\times 10^{-10}$	-2.22045 $\times 10^{-16}$
0.7	8.74819$\times 10^{-10}$	-1.11022 $\times 10^{-15}$
0.8	-0.000021	1.60423 $\times 10^{-11}$
0.9	-0.0000894	-2.33156$\times 10^{-11}$
1	-0.0002041	5.84823 $\times 10^{-11}$

TABLE 1. rror values of $x(t)$ for Example 1.

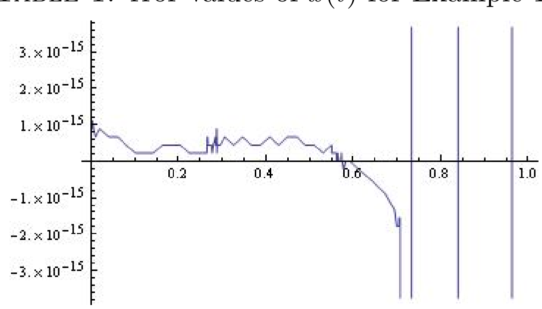

FIGURE 1. Error function of $x(t)$ for $m = 3$ in example 1.

Example 2. Consider the delay differential equation

$$\begin{aligned} \dot{x}(t) &= t^2 x(t-\tfrac{1}{3}), \quad & 0 \le t < 1\,, \\ x(t) &= 0, & -\tfrac{1}{3} \le t < 0\,, \\ x(0) &= 2\,. \end{aligned} \tag{5.5}$$

We approximate the function $\dot{x}(t)$ as

$$\dot{x}(t) = c^T \phi(t), \tag{5.6}$$

so

$$x(t) = c^T \overline{P} \phi(t) + 2 = (c^T \overline{P} + d^T)\phi(t)$$

where $2 = d^T \phi(t)$. Using the delay operational matrix

$$x(t-\frac{1}{3}) = (c^T \overline{P} + d^T)\phi(t-\frac{1}{3}) = (c^T \overline{P} + d^T)(Del)\phi(t). \tag{5.7}$$

If we approximate the function t^2 as

$$t^2 \simeq \phi(t)^T e \tag{5.8}$$

and substitute (5.6), (5.7) and (5.8) in (5.5), we will have

$$c^T \phi(t) = (c^T \overline{P} + d^T)(Del)\phi(t)\phi(t)^T e$$

then

$$c^T = (c^T \overline{P} + d^T)(Del)\widehat{e} \tag{5.9}$$

which $\widehat{e}$ is operational matrix of product that

$$\phi(t)\phi(t)^T e \simeq \widehat{e}\, \phi(t)\,.$$

If we solve the set (5.9) with $m = 6$ and $k = 3$, we will get

$$c_0 = c_1 = c_2 = c_3 = c_4 = c_5 = c_6 = 0,\ c_7 = \frac{2}{9},\ c_8 = \frac{8}{27}$$

$$c_9 = \frac{52}{135},\ c_{10} = \frac{22}{45},\ c_{11} = \frac{82}{135},\ c_{12} = \frac{20}{27},\ c_{13} = \frac{8}{9},\ c_{14} = \frac{8}{9},\ c_{15} = \frac{760}{729}$$

$$c_{16} = \frac{886}{729},\ c_{17} = \frac{10279}{7290},\ c_{18} = \frac{17828}{10935},\ c_{19} = \frac{1372}{729},\ c_{20} = \frac{176}{81}$$

and $c^T = [c_0, c_1, \ldots, c_{20}]$

$$x(t) = c^T \overline{P} \phi(t) + 2$$

$$= \begin{cases} 2 & 0 \le t < \frac{1}{3} \\ \frac{2}{3}t^3 + \frac{160}{81} & \frac{1}{3} \le t < \frac{2}{3} \\ \frac{t^6}{9} - \frac{2}{15}t^5 + \frac{t^4}{18} + \frac{158}{243}t^3 + \frac{64856}{32805} & \frac{2}{3} \le t < 1 \end{cases}$$

which is the exact solution.

Example 3. Consider the following delay system with delay in both control and state [13, 15]

$$\begin{array}{ll} \dot{x}(t)+x(t)+2x(t-\frac{1}{4})=2u(t-\frac{1}{4}) & 0\le t<1, \\ x(t)=u(t)=0, & -\frac{1}{4}\le t\le 0, \\ u(t)=1, & t>0. \end{array} \tag{5.10}$$

The exact solution is [5]

$$x(t)=\begin{cases} 0, & 0\le t<\frac{1}{4}, \\ 2-2e^{-(t-\frac{1}{4})}, & \frac{1}{4}\le t<\frac{1}{2}, \\ -2-2e^{-(t-\frac{1}{4})}+(2+4t)e^{-(t-\frac{1}{2})}, & \frac{1}{2}\le t<\frac{3}{4}, \\ 6-2e^{-(t-\frac{1}{4})}+(2+4t)e^{-(t-\frac{1}{2})}-(\frac{17}{4}+2t+4t^2)e^{-(t-\frac{3}{4})}, & \frac{3}{4}\le t<1\,. \end{cases}$$

Using (5.10) we know

$$u(t)=\begin{cases} 0, & -\frac{1}{4}\le t\le 0, \\ 1, & t>0\,, \end{cases}$$

then

$$u(t-\frac{1}{4})=\begin{cases} 0, & 0\le t\le\frac{1}{4}, \\ 1, & t>\frac{1}{4}, \end{cases} = u^T\phi(t)$$

which the vector u can be calculated by (3.2). Suppose that

$$\dot{x}(t)=c^T\phi(t)$$

so

$$x(t)=c^T\overline{P}\phi(t).$$

where c is unknown vector. Using the delay operational matrix and (5.10)

$$c^T=2u^T(I+\overline{P}+2\overline{P}(Del))^{(-1)} \tag{5.11}$$

where I is identity matrix with appropriated dimension. We solve (5.11) with $m=3$, $k=4$ and $m=6$, $k=4$ and present the error of $x(t)$ for some points in Table 2 and plot the error function of $x(t)$ for $m=6$, $k=4$ in Figure 2. From numerical results, it can be guessed that the method is convergent of order two.

t	$m=3$, $k=4$	$m=6$, $k=4$
0	0	0
0.1	0	0
0.2	0	0
0.3	-1.66361×10^{-6}	2.01217×10^{-12}
0.4	9.18781×10^{-7}	1.83076×10^{-12}
0.5	0.0000268416	7.80909×10^{-11}
0.6	6.50365×10^{-6}	-2.28817×10^{-11}
0.7	-0.0000110432	-2.5155×10^{-11}
0.8	-0.0000252085	1.33697×10^{-10}
0.9	0.0000135575	1.21643×10^{-10}
1	0.0000626953	-4.13983×10^{-10}

TABLE 2. Error of $x(t)$ for Example 3.

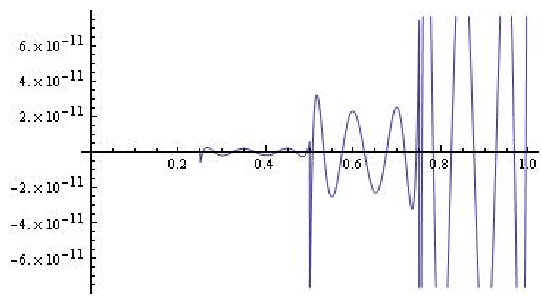

FIGURE 2. Error function of $x(t)$ in Example 3 for $m=6$, $k=4$.

Example 4. Consider the delay differential equation [13]

$$\begin{pmatrix} \dot{x_1}(t) \\ \dot{x_2}(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -25 & -5t \end{pmatrix} \begin{pmatrix} x_1(t-\frac{1}{4}) \\ x_2(t-\frac{1}{4}) \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

with

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \left[-\frac{1}{4}, 0\right].$$

The exact solutions are [8]

$$x_1(t) = \begin{cases} 0, & 0 \le t < \frac{1}{4} \\ \frac{1}{32} - \frac{t}{4} + \frac{t^2}{2}, & \frac{1}{4} \le t < \frac{1}{2} \\ \frac{1}{32} - \frac{19}{96}t + \frac{3}{16}t^2 + \frac{5}{8}t^3 - \frac{5}{12}t^4, & \frac{1}{2} \le t < \frac{3}{4} \\ -\frac{9641}{32768} + \frac{37391}{24576}t - \frac{3183}{1024}t^2 + \frac{785}{256}t^3 - \frac{45}{128}t^4 - \frac{85}{96}t^5 + \frac{5}{18}t^6, & \frac{3}{4} \le t < 1 \end{cases}$$

and

$$x_2(t) = \begin{cases} t, & 0 \le t < \frac{1}{4} \\ -\frac{5}{384} + t + \frac{5}{8}t^2 - \frac{5}{3}t^3, & \frac{1}{4} \le t < \frac{1}{2} \\ \frac{775}{1536} - \frac{17}{8}t + \frac{1295}{192}t^2 - \frac{115}{24}t^3 - \frac{75}{32}t^4 + \frac{5}{3}t^5, & \frac{1}{2} \le t < \frac{3}{4} \\ \frac{3666575}{5505024} - \frac{1051}{1024}t - \frac{95755}{49152}t^2 + \frac{21515}{1536}t^3 - \frac{55325}{3072}t^4 + \frac{335}{96}t^5 + \frac{2125}{576}t^6 - \frac{25}{21}t^7, & \frac{3}{4} \le t < 1\,. \end{cases}$$

We approximate the function $\dot{x_1}(t)$ and $\dot{x_2}(t)$ as

$$\dot{x_1}(t) = c^T\phi(t), \quad \dot{x_2}(t) = u^T\phi(t) \tag{5.12}$$

by initial conditions

$$x_1(t) = c^T\overline{P}\phi(t), \quad x_2(t) = u^T\overline{P}\phi(t).$$

Using the delay operational matrix

$$\begin{aligned} x_1(t-\tfrac{1}{4}) &= c^T\overline{P}\phi(t-\tfrac{1}{4}) = c^T\overline{P}(Del)\phi(t), \\ x_2(t-\tfrac{1}{4}) &= u^T\overline{P}\phi(t-\tfrac{1}{4}) = u^T\overline{P}(Del)\phi(t), \end{aligned} \tag{5.13}$$

If we approximate the functions

$$1 = d^T\phi(t)\ ,\ t = \phi(t)^T e, \tag{5.14}$$

and use the operational matrix of product $\widehat{e}$ and substitute (5.12), (5.13) and (5.14) in main problem, we will have

$$\begin{cases} c^T = u^T\overline{P}(Del), \\ u^T = -25\, c^T\overline{P}(Del) - 5\, u^T\overline{P}(Del)\widehat{e} + d^T, \end{cases} \tag{5.15}$$

which $\widehat{e}$ is the operational matrix of product

$$\phi(t)\phi(t)^T e = \widehat{e}\,\phi(t).$$

Solving (5.15) by $N = k = 4$ and $m = 7$ gives the exact values for $x_1(t)$ and $x_2(t)$.

6. Conclusions

In this paper the operational matrices of integration, dual, differentiation, product and delay for hybrid of block-pulse functions and Bernstein polynomials are obtained. An upper bound for the error of approximation is given. The presented upper bound of error suggests rapidly convergent to the exact solution when $m \to \infty$. The hybrid of block-pulse functions and Bernstein polynomials are used to solve delay differential systems and delay differential equations. The problem has been reduced to solve a set of algebraic equations. It is also shown that the hybrid of block-pulse functions and Bernstein polynomials provide an exact solution when the exact solutions are piecewise polynomial. The illustrative examples demonstrate that the proposed method is valid.

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