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An analytic study on the Euler-Lagrange equation arising in calculus of variations

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Abstract The Euler-Lagrange equation plays an important role in the minimization problems of the calculus of variations. This paper employs the differential transformation method (DTM) for finding the solution of the Euler-Lagrange equation which arise from problems of calculus of variations. DTM provides an analytical solution in the form of an infinite power series with easily computable components. Several illustrative examples are given to demonstrate the effectiveness of the present method.

Keywords. Differential transformation method, Calculus of variation, Euler-Lagrange equation, Variational problems.

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1. Introduction

Variational problems, appear in engineering and science where minimization of functionals, such as Lagrangian, strain, potential, total energy, etc., give the laws governing the systems behavior. Historical comments about variational problems are found in [11]. Also fractional variational problems [2, 18] and fractional boundary value problems [8, 9, 22] have gained considerable importance during the last decade. Several methods have been used to solve variational problems. The well-known direct method of Ritz and Galerkin in solving variational problems has been of considerable concern and is well covered in many textbooks. Walsh series [6], Chebyshev series [14], Legendre series [4], Legendre wavelets [21], rationalized Haar functions [20] and Haar orthonormal wavelet [15] are applied on variational problems. In recent years, some

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researchers for the variational problems have studied on numerical methods such as hybrid of block-pulse and Lagrange interpolating [17], nonclassical parameterization [16], Chebyshev finite difference method [23], homotopy-perturbation method [1], Adomian decomposition method [7] and variational iteration method [24]. In this paper, a relatively new computed approach called the differential transformation method (DTM) was introduced to find the solution of differential equations which arise from problems of calculus of variations. DTM is a semi-numerical-analytic technique that formalizes the Taylor series in a totally different manner. This method was introduced firstly by Zhou to study electrical circuits [25]. DTM constructs an analytical solution in the form of a polynomial and the solution is rapid convergence to an accurate solution [5, 25]. This method gives exact values of the nth derivative of an analytic function at a point in terms of known and unknown boundary conditions in a fast manner [13]. As said in [10], the main advantage of DTM is that it can be applied directly to nonlinear ordinary and partial differential equations without requiring linearization, discretization or perturbation. Also another important advantage is that it can be computerized to greatly reduce the size of computational work. Status of the differential transformation method has been discussed in [3]. DTM has been used to solve effectively, easily and accurately a large class of linear and nonlinear problems with approximations. Today, there are many works on DTM (see for example [3, 10, 12, 13, 19]).

2. Fundamentals of differential transformation method

With reference to the articles [3, 5, 10, 12, 13, 19, 25], the basic definition of the differential transformation are introduced as follows:

Let $x(t)$ be analytic in a domain D and let t_i represent any point in D. The function $x(t)$ is then represented by one power series whose center is located at t_0. As explained in [25] the kth differential transformation of the function $x(t)$ is defined as follows:

$$X(k) = \frac{1}{k!}\left[\frac{d^k x(t)}{dt^k}\right]_{t=t_0}, \qquad \forall t \in D. \tag{2.1}$$

In Eq. (2.1), $x(t)$ is the original function and $X(k)$ is the transformed function. The inverse differential transformation is given by

$$x(t) = \sum_{k=0}^{\infty} (t-t_0)^k X(k), \qquad \forall t \in D. \tag{2.2}$$

Combining Eqs. (2.1) and (2.2), we have

$$x(t) = \sum_{k=0}^{\infty} \frac{(t-t_0)^k}{k!} \left[\frac{d^k x(t)}{dt^k}\right]_{t=t_0}, \qquad \forall t \in D. \tag{2.3}$$

Eq. (2.3) implies that the concept of the differential transformation is derived from Taylor's series expansion, but this method does not evaluate the derivatives symbolically. Some of the fundamental mathematical operations performed by differential transform method are listed in Table 1.

Table 1. The fundamental operations of DTM.

Original function	Transformed function
$x(t) = \alpha u(t) \pm \beta w(t)$	$X(k) = \alpha U(k) \pm \beta W(k)$
$x(t) = u(t)v(t)$	$X(k) = \sum_{\ell=0}^{k} U(\ell)V(k-\ell)$
$x(t) = \frac{d^m}{dt^m} u(t)$	$X(k) = \frac{(k+m)!}{k!} U(k+m)$
$x(t) = t^m$	$X(k) = \delta(k-m) = \begin{cases} 1, & k=m, \\ 0, & k \neq m, \end{cases}$
$x(t) = e^{mt}$	$X(k) = \frac{m^k}{k!}$
$x(t) = u_1(t)u_2(t)...u_{m-1}(t)u_m(t)$	$Z(k) = \sum_{\ell_{m-1}=0}^{k} \sum_{\ell_{m-2}=0}^{\ell_{m-1}} \cdots \sum_{\ell_2=0}^{\ell_3} \sum_{\ell_1=0}^{\ell_2} \{U_1(\ell_1)$
	$U_2(\ell_2-\ell_1)...U_{m-1}(\ell_{m-1}-\ell_{m-2})U_m(k-\ell_{m-1})\}$

3. Description of the problem

Consider the problem of finding the extremum of the functional

$$J[y(x)] = \int_{x_0}^{x_1} F(x, y(x), y'(x))dx. \tag{3.1}$$

The necessary condition for $y(x)$ to extremize $J[y(x)]$ is that it should satisfy the *Euler-Lagrange equation*

$$F_y - \frac{d}{dx} F_{y'} = 0, \tag{3.2}$$

with appropriate boundary conditions. The boundary value problem (3.2) does not always have a solution and if the solution exists, it may not be unique. It is worth to mention here that, if the solution of Euler-Lagrange satisfies the boundary conditions, it is unique. The general form of (3.1) is finding the extremum of the functional

$$J[y_1, y_2, ..., y_n] = \int_{x_0}^{x_1} F(x, y_1, y_2, ..., y_n, y_1', y_2', ..., y_n')dx, \tag{3.3}$$

with the given boundary conditions for all functions

$$y_i(x_0) = \alpha_i, \qquad i = 1, 2, ..., n, \tag{3.4}$$

$$y_i(x_1) = \beta_i, \qquad i = 1, 2, ..., n. \tag{3.5}$$

The necessary condition for $y_i(x)$, $i = 1, 2, ..., n$, to extremize functional (3.3) is to satisfy the following system of second-order differential equations

$$F_{y_i} - \frac{d}{dx} F_{y_i'} = 0, \qquad\qquad i = 1, 2, ..., n, \tag{3.6}$$

with boundary conditions given in (3.4) and (3.5). Also it is possible to define the variational problem for functionals dependent on higher-order derivatives in the following form [11]

$$J[y] = \int_{x_0}^{x_1} F\left(x, y, y', ..., y^{(n)}\right) dx, \tag{3.7}$$

where we consider the function F differentiable $n+2$ times with respect to all arguments and we assume that the boundary conditions are of the form

$$\begin{aligned} &y(x_0) = y_0,\ y'(x_0) = y_0',\ ...,\ y^{(n-1)}(x_0) = y_0^{(n-1)}, \\ &y(x_1) = y_1,\ y'(x_1) = y_1',\ ...,\ y^{(n-1)}(x_1) = y_1^{(n-1)}. \end{aligned} \tag{3.8}$$

The necessary condition for the extremum of the functional (3.7) is to satisfy the following differential equation of order $2n$

$$F_y - \frac{d}{dx} F_{y'} + \frac{d^2}{dx^2} F_{y''} + ... + (-1)^n \frac{d^n}{dx^n} F_{y^{(n)}} = 0. \tag{3.9}$$

This equation is called the *Euler-Poisson equation*.

4. Solution with DTM

To show the efficiency of the DTM method described above, we present some examples.

Example 1. Consider the following variational problem [7, 23, 24]:

$$\min J = \int_0^1 [y(x) + y'(x) - 4\exp(3x)]^2 dx, \tag{4.1}$$

with the given boundary conditions

$$y(0) = 1, \qquad\qquad y(1) = \exp(3). \tag{4.2}$$

The corresponding Euler-Lagrange equation is

$$y'' - y - 8\exp(3x) = 0, \tag{4.3}$$

with boundary conditions (4.2). Taking differential transform of Eq. (4.3), one can obtain

$$(k+1)(k+2)Y(k+2) - Y(k) - 8\frac{3^k}{k!} = 0, \tag{4.4}$$

where $Y(k)$ is the differential transformations of function $y(x)$. From the boundary conditions (4.2) we can write

$$Y(0) = 1, \tag{4.5}$$

$$\sum_{k=0}^{\infty} Y(k) = \exp(3). \tag{4.6}$$

Using Eqs. (4.4), and (4.5) for $k = 0, 2, 4, ...$ we get

$$Y(2) = \frac{3^2}{2!}, \quad Y(4) = \frac{3^4}{4!}, \quad Y(6) = \frac{3^6}{6!}, \quad Y(8) = \frac{3^8}{8!}, \cdots .$$

It can be easily shown that

$$Y(k) = \frac{3^k}{k!}, \qquad k = 0, 2, 4, \cdots . \tag{4.7}$$

Also, using Eq. (4.2), for $k = 3, 5, 7, ...$ we get

$$\begin{aligned}
Y(3) &= \frac{1}{3!}Y(1) + \frac{8}{3!}(3), \\
Y(5) &= \frac{1}{5!}Y(1) + \frac{8}{5!}(3 + 3^3), \\
Y(7) &= \frac{1}{7!}Y(1) + \frac{8}{7!}(3 + 3^3 + 3^5), \\
Y(9) &= \frac{1}{9!}Y(1) + \frac{8}{9!}(3 + 3^3 + 3^5 + 3^7), \\
&\vdots \\
Y(k) &= \frac{1}{k!}Y(1) + \frac{8}{k!}(3 + 3^3 + ... + 3^{k-2}), \qquad k = 3, 5, 7, ...
\end{aligned}$$

But $3 + 3^3 + ... + 3^{k-2} = (3^k - 3)/8$, thus

$$Y(k) = \frac{1}{k!}Y(1) + \frac{3^k - 3}{k!}, \qquad k = 3, 5, 7, ... \tag{4.8}$$

Employing Eqs. (4.6),(4.7) and (4.8) we obtain

$$\sum_{\substack{k=0\\(k)\ even}}^{\infty}\frac{3^k}{k!}+Y(1)+\sum_{\substack{k=3\\(k)\ odd}}^{\infty}\frac{3^k}{k!}+\sum_{\substack{k=3\\(k)\ odd}}^{\infty}\frac{(Y(1)-3)}{k!}=\exp(3),$$

therefore

$$(Y(1)-3)\left(1+\sum_{\substack{k=3\\(k)\ odd}}^{\infty}\frac{1}{k!}\right)=0.$$

Consequently $Y(1)=3$. Now by using Eqs. (4.7) and (4.8) we get

$$Y(k)=\frac{3^k}{k!},\qquad k=0,1,2,... \tag{4.9}$$

Thus

$$y(x)=\sum_{k=0}^{\infty}x^kY(k)=\sum_{k=0}^{\infty}\frac{(3x)^k}{k!}=\exp(3x),$$

which is the exact solution.

Example 2. Consider the following variational problem

$$\min J=\int_0^{\frac{\pi}{2}}[y''^2(x)-y^2(x)+x^2]dx, \tag{4.10}$$

that satisfies the conditions

$$y(0)=1,\quad y'(0)=0,\quad y\left(\frac{\pi}{2}\right)=0,\quad y'\left(\frac{\pi}{2}\right)=-1. \tag{4.11}$$

The corresponding Euler-Poisson equation is

$$y^{(4)}-y=0. \tag{4.12}$$

Taking the differential transform of Eq. (4.12), one can obtain

$$(k+1)(k+2)(k+3)(k+4)Y(k+4)-Y(k)=0. \tag{4.13}$$

From the boundary conditions (4.11) we can write

$$Y(0)=1,\qquad Y(1)=0, \tag{4.14}$$

$$\sum_{k=0}^{\infty}\left(\frac{\pi}{2}\right)^kY(k)=0,\qquad \sum_{k=0}^{\infty}(k+1)\left(\frac{\pi}{2}\right)^kY(k+1)=-1. \tag{4.15}$$

For $k = 0, 1, 2, \cdots$ it can be easily shown that

$$\begin{aligned}
&Y(4) = \tfrac{1}{4!}, && Y(8) = \frac{1}{8!}, && \cdots, \ Y(4k) = \frac{1}{(4k)!},\\
&Y(5) = 0, && Y(9) = 0, && \cdots, \ Y(4k+1) = 0,\\
&Y(6) = \tfrac{2}{6!}Y(2), && Y(10) = \frac{2}{10!}Y(2), && \cdots, \ Y(4k+2) = \frac{2}{(4k+2)!}Y(2),\\
&Y(7) = \tfrac{6}{7!}Y(3), && Y(11) = \frac{6}{11!}Y(3), && \cdots, \ Y(4k+3) = \frac{6}{(4k+3)!}Y(3).
\end{aligned}$$

Now substituting above expressions into Eq. (4.15), the following system is obtained

$$\begin{cases}
1 + \left(\frac{\pi}{2}\right)^2 Y(2) + \left(\frac{\pi}{2}\right)^3 Y(3) + \left(\frac{1}{4!}\right)\left(\frac{\pi}{2}\right)^4 + 2\left(\frac{1}{6!}\right)\left(\frac{\pi}{2}\right)^6 Y(2) + 6\left(\frac{1}{7!}\right)\left(\frac{\pi}{2}\right)^7 Y(3)\\
+ \left(\frac{1}{8!}\right)\left(\frac{\pi}{2}\right)^8 + \cdots = 0,\\
2\left(\frac{\pi}{2}\right) Y(2) + 3\left(\frac{\pi}{2}\right)^2 Y(3) + 2\left(\frac{1}{5!}\right)\left(\frac{\pi}{2}\right)^5 Y(2) + 6\left(\frac{1}{6!}\right)\left(\frac{\pi}{2}\right)^6 Y(3) + \cdots =\\
-\left[1 + \left(\frac{1}{3!}\right)\left(\frac{\pi}{2}\right)^3 + \left(\frac{1}{7!}\right)\left(\frac{\pi}{2}\right)^7 + \cdots\right].
\end{cases} \tag{4.16}$$

By simplifying Eq. (4.16), we have

$$\begin{cases}
2Y(2)\left[\frac{1}{2!}\left(\frac{\pi}{2}\right)^2 + \frac{1}{6!}\left(\frac{\pi}{2}\right)^6 + \cdots\right] + 6Y(3)\left[\frac{1}{3!}\left(\frac{\pi}{2}\right)^3 + \frac{1}{7!}\left(\frac{\pi}{2}\right)^7 + \cdots\right] =\\
-\left[1 + \frac{1}{4!}\left(\frac{\pi}{2}\right)^4 + \frac{1}{8!}\left(\frac{\pi}{2}\right)^8 + \cdots\right],\\
2Y(2)\left[\left(\frac{\pi}{2}\right) + \frac{1}{5!}\left(\frac{\pi}{2}\right)^5 + \cdots\right] + 6Y(3)\left[\frac{1}{2!}\left(\frac{\pi}{2}\right)^2 + \frac{1}{6!}\left(\frac{\pi}{2}\right)^6 + \cdots\right] =\\
-\left[1 + \frac{1}{3!}\left(\frac{\pi}{2}\right)^3 + \frac{1}{7!}\left(\frac{\pi}{2}\right)^7 + \cdots\right].
\end{cases} \tag{4.17}$$

From Taylor series for functions $\sin(x)$ and $\cos(x)$ at $x = \pi/2$, we get

$$1 + \frac{1}{3!}\left(\frac{\pi}{2}\right)^3 + \frac{1}{7!}\left(\frac{\pi}{2}\right)^7 + \cdots = \left(\frac{\pi}{2}\right) + \frac{1}{5!}\left(\frac{\pi}{2}\right)^5 + \cdots \tag{4.18}$$

$$\frac{1}{2!}\left(\frac{\pi}{2}\right)^2 + \frac{1}{6!}\left(\frac{\pi}{2}\right)^6 + \cdots = 1 + \frac{1}{4!}\left(\frac{\pi}{2}\right)^4 + \frac{1}{8!}\left(\frac{\pi}{2}\right)^8 + \cdots \tag{4.19}$$

respectively. Substituting Eqs. (4.18) and (4.19) into Eq. (4.17) we have

$$\begin{cases} 2Y(2)\Big[\frac{1}{2!}\left(\frac{\pi}{2}\right)^2+\frac{1}{6!}\left(\frac{\pi}{2}\right)^6+\cdots\Big]+6Y(3)\Big[-1+\left(\frac{\pi}{2}\right)+\frac{1}{5!}\left(\frac{\pi}{2}\right)^5+\cdots\Big]= \\ -\Big[\frac{1}{2!}\left(\frac{\pi}{2}\right)^2+\frac{1}{6!}\left(\frac{\pi}{2}\right)^6+\cdots\Big], \\ 2Y(2)\left[\left(\frac{\pi}{2}\right)+\frac{1}{5!}\left(\frac{\pi}{2}\right)^5+\cdots\right]+6Y(3)\Big[\frac{1}{2!}\left(\frac{\pi}{2}\right)^2+\frac{1}{6!}\left(\frac{\pi}{2}\right)^6+\cdots\Big]= \\ -\left[\left(\frac{\pi}{2}\right)+\frac{1}{5!}\left(\frac{\pi}{2}\right)^5+\cdots\right]. \end{cases} \tag{4.20}$$

Using Gaussian elimination, if we multiply the first equation by $-\left[\frac{1}{2!}(\frac{\pi}{2})^2+\frac{1}{6!}(\frac{\pi}{2})^6+\cdots\right]$ and the second equation by $\left[-1+\left(\frac{\pi}{2}\right)+\left(\frac{1}{5!}\right)\left(\frac{\pi}{2}\right)^5+\cdots\right]$, simply by adding two equations we obtain

$$Y(2)=-0.5, \tag{4.21}$$

substituting Eq. (4.21) into Eq. (4.20) we obtain

$$Y(3)=0. \tag{4.22}$$

Now, using Eqs. (4.21) and (4.22) we get

$$Y(k)=\begin{cases} \frac{1}{k!}, & \text{for } k=0,4,8,..., \\ 0, & \text{for } k=1,3,5,... \\ \frac{-1}{k!}, & \text{for } k=2,6,10,... \end{cases} \tag{4.23}$$

thus

$$y(x) \;=\; \sum_{k=0}^{\infty} x^k Y(k)=\sum_{k=0}^{\infty}\frac{(-1)^k x^{(2k)}}{(2k)!}=\cos(x),$$

which is the exact solution.

Example 3. In this example, we consider the following variational problem [23, 24]

$$\min J=\int_0^1 \frac{1+y^2(x)}{y'^2(x)}dx. \tag{4.24}$$

Let the boundary conditions be

$$y(0)=0, \qquad y(1)=0.5, \tag{4.25}$$

in this case the Euler-Lagrange equation is written in the following form:

$$y'' + y''y^2 - yy'^2 = 0. \tag{4.26}$$

Taking the differential transform of Eq. (4.26), we can obtain

$$(k+1)(k+2)Y(k+2) + \sum_{\ell=0}^{k}\sum_{m=0}^{\ell}(k-\ell+1)(k-\ell+2)Y(m)Y(\ell-m)Y(k-\ell+2) -$$
$$\sum_{\ell=0}^{k}\sum_{m=0}^{\ell}(m+1)(\ell-m+1)Y(m+1)Y(\ell-m+1)Y(k-\ell) = 0.$$

From the boundary conditions (4.25) we obtain

$$Y(0) = 0, \tag{4.27}$$

$$\sum_{k=0}^{\infty} Y(k) = 0.5, \tag{4.28}$$

using Eqs. (4.27) and (4.27) we get

$$Y(k) = \begin{cases} 0, & \text{for } k = 0, 2, 4, ..., \\ \frac{1}{k!}[Y(1)]^k, & \text{for } k = 1, 3, 5, ... \end{cases} \tag{4.29}$$

Substituting Eq. (4.29) into Eq. (4.28) we have

$$\sum_{k=0}^{\infty} \frac{[Y(1)]^{2k+1}}{(2k+1)!} = 0.5, \qquad \text{or} \qquad \sinh(Y(1)) = 0.5, \tag{4.30}$$

by solving this equation we get $Y(1) = 0.4812118250$. Thus

$$y(x) = \sum_{k=0}^{\infty} x^k Y(k) = \sum_{k=0}^{\infty} \frac{(0.4812118250x)^{2k+1}}{(2k+1)!}$$
$$= \sinh(0.4812118250x),$$

which is the exact solution.

Example 4. Consider the problem of finding the extremals of the functional [23, 24].

$$J(y(x), z(x)) = \int_0^{\frac{\pi}{2}} [y'^2(x) + z'^2(x) + 2y(x)z(x)]dx, \tag{4.31}$$

that satisfies the conditions

$$\begin{cases} y(0) = 0, & y(\frac{\pi}{2}) = 1, \\ z(0) = 0, & z(\frac{\pi}{2}) = -1. \end{cases} \tag{4.32}$$

The system of Euler's differential equations is of the form:

$$\begin{cases} y''(x) - z(x) = 0, \\ z''(x) - y(x) = 0. \end{cases} \tag{4.33}$$

Taking the differential transform of Eq. (4.33), we can obtain

$$\begin{cases} (k+1)(k+2)Y(k+2) - Z(k) = 0, \\ (k+1)(k+2)Z(k+2) - Y(k) = 0, \end{cases} \tag{4.34}$$

where $Y(k)$ and $Z(k)$ are the differential transformations of functions $y(x)$ and $z(x)$ respectively. From the boundary conditions (4.32) we can write

$$Y(0) = 0, \qquad Z(0) = 0, \tag{4.35}$$

$$\sum_{k=0}^{\infty} \left(\frac{\pi}{2}\right)^k Y(k) = 1, \qquad \sum_{k=0}^{\infty} \left(\frac{\pi}{2}\right)^k Z(k) = -1. \tag{4.36}$$

By using Eqs. (4.34) and (4.35), it can be easily shown that

$$Y(2k) = Z(2k) = 0, \qquad k = 0, 1, 2, ... \tag{4.37}$$

$$Y(2k+1) = \begin{cases} \frac{1}{(2k+1)!} Z(1), & \text{for } k = 1, 3, 5, ... \\ \frac{1}{(2k+1)!} Y(1), & \text{for } k = 2, 4, 6, ... \end{cases}, \tag{4.38}$$

$$Z(2k+1) = \begin{cases} \frac{1}{(2k+1)!} Y(1), & \text{for } k = 1, 3, 5, ... \\ \frac{1}{(2k+1)!} Z(1), & \text{for } k = 2, 4, 6, ... \end{cases} \tag{4.39}$$

Substituting Eqs. (4.37), (4.38) and (4.39) into Eq. (4.36) we obtain

$$\left(\frac{\pi}{2}\right) Y(1) + Y(1) \sum_{\substack{k=2 \\ (k)\ even}}^{\infty} \frac{\left(\frac{\pi}{2}\right)^{(2k+1)}}{(2k+1)!} + Z(1) \sum_{\substack{k=1 \\ (k)\ odd}}^{\infty} \frac{\left(\frac{\pi}{2}\right)^{(2k+1)}}{(2k+1)!} = 1, \tag{4.40}$$

$$\left(\frac{\pi}{2}\right) Z(1) + Z(1) \sum_{\substack{k=2 \\ (k)\ even}}^{\infty} \frac{\left(\frac{\pi}{2}\right)^{(2k+1)}}{(2k+1)!} + Y(1) \sum_{\substack{k=1 \\ (k)\ odd}}^{\infty} \frac{\left(\frac{\pi}{2}\right)^{(2k+1)}}{(2k+1)!} = -1, \tag{4.41}$$

simply by adding Eqs. (4.40) and (4.41) we have

$$(Y(1) + Z(1)) \sum_{k=0}^{\infty} \frac{\left(\frac{\pi}{2}\right)^{(2k+1)}}{(2k+1)!} = 0, \qquad \text{or} \qquad (Y(1) + Z(1)) \sinh\left(\frac{\pi}{2}\right) = 0.$$

Thus $Z(1) = -Y(1)$. Now using Eq. (4.40) we get

$$Y(1)\sum_{k=0}^{\infty}\frac{(-1)^k\left(\frac{\pi}{2}\right)^{(2k+1)}}{(2k+1)!} = 1, \qquad \text{or} \qquad Y(1)\sin\left(\frac{\pi}{2}\right) = 1.$$

Therefore

$$Y(1) = 1, \qquad Z(1) = -1. \tag{4.42}$$

Finally by using Eqs. (4.37),(4.38) and (4.42) we obtain

$$\begin{aligned} y(x) &= \sum_{k=0}^{\infty} x^k Y(k) = \sum_{k=0}^{\infty}\frac{(-1)^k x^{(2k+1)}}{(2k+1)!} = \sin(x), \\ z(x) &= \sum_{k=0}^{\infty} x^k Z(k) = -\sum_{k=0}^{\infty}\frac{(-1)^k x^{(2k+1)}}{(2k+1)!} = -\sin(x), \end{aligned}$$

which is the exact solution.

5. Conclusion

In the present work, the differential transformation method is used for finding the minimum of a functional over the specified domain. The main objective is to find the solution of an ordinary differential equation which arises from the variational problem. The method is characterized by simplicity, efficiency and it is also readily implemented. Several examples are given and the results demonstrate the reliability and efficiency of the method for solving this type of problem.

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