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Inverse Sturm-Liouville problems with transmission and spectral parameter boundary conditions

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Abstract This paper deals with the boundary value problem involving the differential equation

$$\ell y := -y'' + qy = \lambda y,$$

subject to the eigenparameter dependent boundary conditions along with the following discontinuity conditions

$$y(d+0) = ay(d-0), \quad y'(d+0) = ay'(d-0) + by(d-0).$$

In this problem $q(x)$, d, a, b are real, $q \in L^2(0,\pi)$, $d \in (0,\pi)$ and λ is a parameter independent of x. By defining a new Hilbert space and using spectral data of a kind, it is developed the Hochestadt's result based on transformation operator for inverse Sturm-Liouville problem with parameter dependent boundary and discontinuous conditions. Furthermore, it is established a formula for $q(x) - \tilde{q}(x)$ in the finite interval, where $\tilde{q}(x)$ is an analogous function with $q(x)$.

Keywords. Inverse Sturm-Liouville problem; Jump conditions; Green's function; Eigenparameter dependent condition; Transformation operator.

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1. Introduction

We consider the boundary value problem

$$\ell y := -y'' + qy = \lambda y, \tag{1.1}$$

subject to the parameter dependent boundary conditions

$$U(y) := \lambda(y'(0) + h_1 y(0)) - h_2 y'(0) - h_3 y(0) = 0,$$
$$V(y) := \lambda(y'(\pi) + H_1 y(\pi)) - H_2 y'(\pi) - H_3 y(\pi) = 0, \tag{1.2}$$

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and the jump conditions

$$\begin{aligned} &U_1(y) := y(d+0) - ay(d-0) = 0, \\ &U_2(y) := y'(d+0) - ay'(d-0) - by(d-0) = 0, \end{aligned} \tag{1.3}$$

where $q(x)$ is real function in $\in L^2[0,\pi]$, $h_i,\ H_i$, $(i = 1,2,3)$, a, b, and d are real with $d \in (0,\pi)$. $r_1 := h_3 - h_1h_2 > 0$ and $r_2 := H_1H_2 - H_3 > 0$. For simplicity we use the notation $L = L(q; h_i; H_i; d)$ for the problem (1.1)–(1.3). Here λ is the spectral parameter.

In this paper, we study the inverse Sturm-Liouville problems. The inverse Sturm-Liouville problems can be regarded as three aspects, e.g., existence, uniqueness and reconstruction of the potential function q from given spectral data. These problems originated in the work of Ambarzumian(1929) [3], were continued by Borg(1945) [7], and have been gradually elucidated over the past seventy years. Here we want to look at the question of uniqueness for the above problem using two set of spectra, or one spectrum plus part of a set of value of eigenfunctions at some interior point. Such kind of problems have a long tradition and we refer the reader to [3–7], [9–16], [18–22], [25, 27, 29, 31, 32], [34–38], and the references therein. In particular, the operator ℓ plays an important role as the one-dimensional Schrödinger operator in quantum mechanics and our transmission conditions include the case of point interactions (see e.g. the monographs [2, 33]). In this manuscript, we generalize the Hochstadt's result [13], refining the approach of Levinson [25] for eigenparameter dependent boundary conditions for Sturm-Liouville operator to show that precisely how much q has freedom where the λ'_n and all but finitely many of the λ_n are specified. Note that the eigenvalues λ'_n is obtained with replacing H_i by $\mathfrak{H}_i$ in (1.2). There are many papers concerning problems with discontinuous conditions. One can find the similar works for discontinuous conditions in [4, 12, 17, 18, 32, 34–36, 38]. The similar works for Hochstadt's result in [6, 20, 21, 30]. Nowadays there are several number of papers devoted to inverse problems for the Sturm-Liouville operator with eigenparameter dependent boundary conditions in [5, 11, 17, 30, 36, 37].

In section 2 we define a new Hilbert space for the eigenparameter dependent boundary conditions for the Sturm-Liouville operator by using similar techniques as in [1,28], to obtain the asymptotic form of solutions and eigenvalues. In section 3 we formulate a novel inverse Sturm-Liouville problem based on transformation operator.

2. The Hilbert space formulation and asymptotic form of solutions and eigenvalues

In this section, we introduce the special inner product in the Hilbert space $(L_2(0,d)\oplus L_2(d,\pi))\oplus\mathbb{C}^2$ and we define a linear operator A in it such that the problem (1.1)–(1.3) can be interpreted as the eigenvalue problem of A. So, we define a new Hilbert space inner product on $\mathcal{H} := (L_2(0,d)\oplus L_2(d,\pi))\oplus\mathbb{C}^2$ by

$$\langle F,G\rangle_{\mathcal{H}} := |a|\int_0^{d-0} f\bar{g} + \frac{1}{|a|}\int_{d+0}^{\pi} f\bar{g} + \frac{|a|}{r_1} f_1\bar{g}_1 + \frac{1}{r_2|a|} f_2\bar{g}_2, \tag{2.1}$$

where $F(x) = \begin{pmatrix} f(x) \\ f_1 \\ f_2 \end{pmatrix}$ and $G(x) = \begin{pmatrix} g(x) \\ g_1 \\ g_2 \end{pmatrix} \in \mathcal{H}$ and we let

$$R_1(u) := u'(0) + h_1u(0), \qquad R_1'(u) := h_2u(0) + h_3u'(0),$$

$$R_2(u) := u'(\pi) + H_1u(\pi), \qquad R_2'(u) := H_2u(\pi) + H_3u'(\pi).$$

In this Hilbert space we construct the operator

$$A : \mathcal{H} \to \mathcal{H}, \tag{2.2}$$

with domain

$$D(A) = \left\{ \begin{array}{c} F = \begin{pmatrix} f(x) \\ f_1 \\ f_2 \end{pmatrix} \Bigg| f(x), f'(x) \in AC[0,d)\cup(d,\pi] \text{ and}, \\ f(d\pm 0),\ f'(d\pm 0) \text{ is defined},\ \ell f \in L^2[(0,d)\cup(d,\pi)] \\ U(f) = U_1(f) = U_2(f) = 0,\ f_1 = R_1(f),\ f_2 = R_2(f) \end{array} \right\}, \tag{2.3}$$

by action law

$$AF = \begin{pmatrix} \ell f \\ R_1'(f) \\ R_2'(f) \end{pmatrix} \quad \text{with } F = \begin{pmatrix} f(x) \\ R_1(f) \\ R_2(f) \end{pmatrix} \in D(A),$$

thus, we can change the boundary value problem (1.1)-(1.3) as following form

$$AY = \lambda Y, \quad Y := \begin{pmatrix} y(x) \\ R_1(y) \\ R_2(y) \end{pmatrix} \in D(A), \tag{2.4}$$

in the Hilbert space $\mathcal{H}$. It is easy to verify that the eigenvalues of the operator A coincide with those of the problem (1.1)-(1.3).

Theorem 2.1. *The operator A is self-adjoint.*

Proof. We omit the proof, since the arguments are the same as in [1, 28]. $\square$

Suppose that the functions $\varphi(x,\lambda)$ and $\psi(x,\lambda)$ are solutions of (1.1) under the initial conditions

$$\varphi(0,\lambda) = h_2 - \lambda,\ \varphi'(0,\lambda) = \lambda h_1 - h_3, \tag{2.5}$$

and

$$\psi(\pi,\lambda) = H_2 - \lambda,\ \psi'(\pi,\lambda) = \lambda H_1 - H_3, \tag{2.6}$$

and the jump conditions (1.3). By attaching a subscript 1 or 2 to the functions φ and ψ, we mean to refer to the first subinterval $[0,d)$ or to the second subinterval $(d,\pi]$. By virtue of [1] problem (1.1) under the initial conditions (2.5) or (2.6) has a unique solution $\varphi_1(x,\lambda)$ or $\psi_2(x,\lambda)$, which is an entire function of $\lambda \in \mathbb{C}$ for each fixed point $x \in [0,d)$ or $x \in (d,\pi]$. From the linear differential equations we obtain the Wronskians

$$\Delta_1(\lambda) := W(\varphi_1(x,\lambda), \psi_1(x,\lambda)), \tag{2.7}$$

and

$$\Delta_2(\lambda) := W(\varphi_2(x,\lambda), \psi_2(x,\lambda)), \tag{2.8}$$

are independent on $x \in [0,d) \cup (d,\pi]$. By using the jump conditions we obtain $\Delta_2(\lambda) = a^2\Delta_1(\lambda)$, for each $\lambda \in \mathbb{C}$.

Corollary 2.2. *The zeros of $\Delta(\lambda) := \Delta_2(\lambda) = a^2\Delta_1(\lambda)$ coincide, and the eigenvalues of the problem with the zeros (1.1)–(1.3) coincide with the zeros of the function $\Delta(\lambda)$.*

Corollary 2.3. *By self-adjointness of A and Corollary 2.2, all eigenvalues of the problem (1.1)–(1.3) are real and simple.*

Theorem 2.4. *Let $\lambda = \rho^2$ and $\tau := \mathrm{Im}\rho$. For equation (1.1) with spectral parameter dependent boundary conditions (1.2) and jump conditions (1.3) as $|\lambda| \to \infty$, the following asymptotic formulas hold:*

$$\varphi(x;\lambda) = \begin{cases} \rho^2 \cos\rho x + \rho(-h_1 + \frac{1}{2}\int_0^x q(t)dt)\sin\rho x + O(\exp(|\tau|x)), & x < d, \\ \\ a\rho^2 \cos\rho x + \rho(f_1(x)\sin\rho x + f_2(x)\sin\rho(2d-x)) \\ \qquad\qquad +O(\exp(|\tau|x)), & x > d, \end{cases} \tag{2.9}$$

$$\varphi'(x;\lambda)=\begin{cases} -\rho^3\sin\rho x+\rho^2(-h_1+\frac{1}{2}\int_0^x q(t)dt)\cos\rho x \\ \qquad\qquad\qquad +O(\rho\exp(|\tau|x)), \qquad x<d, \\ -a\rho^3\sin\rho x+\rho^2(f_1(x)\cos\rho x \\ \qquad -f_2(x)\cos\rho(2d-x))+O(\rho\exp(|\tau|x)), \quad x>d, \end{cases} \tag{2.10}$$

and

$$\psi(x;\lambda)=\begin{cases} \frac{1}{a}\rho^2\cos\rho(\pi-x)+\rho\,(g_1(x)\sin\rho(\pi-x) \\ \qquad +g_2(x)\sin\rho(2d+x-\pi))+O(\exp|\tau|(\pi-x)), \quad x<d, \\ \\ \rho^2\cos\rho(\pi-x)+\rho(H_1+\frac{1}{2}\int_{\pi-x}^{\pi} q(x)dx)\sin\rho(\pi-x) \\ \qquad\qquad\qquad +O(\exp|\tau|(\pi-x)), \qquad x>d, \end{cases} \tag{2.11}$$

$$\psi'(x;\lambda)=\begin{cases} \frac{1}{a}\rho^3\sin\rho(\pi-x)+\rho^2(-g_1(x)\cos\rho(\pi-x) \\ \qquad +g_2(x)\cos\rho(2d+x-\pi))+O(\rho\exp(|\tau|\pi-x)), \quad x<d, \\ \\ \rho^3\sin\rho(\pi-x)-\rho^2(H_1+\frac{1}{2}\int_{\pi-x}^{\pi} q(x)dx)\cos\rho(\pi-x) \\ \qquad\qquad\qquad +O(\rho\exp|\tau|(\pi-x)), \qquad x>d, \end{cases} \tag{2.12}$$

where

$$f_1(x)=a\left(-h_1+\frac{1}{2}\int_0^x q(t)dt\right)+\frac{b}{2}, \quad f_2(x)=\frac{b}{2},$$

$$g_1(x)=\frac{1}{a}\left(H_1+\frac{1}{2}\int_{\pi-x}^{\pi} q(t)dt\right)-\frac{b}{2a^2}, \quad g_2(x)=-\frac{b}{2a^2}.$$

The characteristic function is

$$\begin{aligned}\Delta(\lambda)=&-a\rho^5\sin\rho\pi+\rho^4[(f_1(\pi)+aH_1)\cos\rho\pi-f_2(\pi)\cos\rho(2d-\pi)] \\ &+O(\rho^3\exp(|\tau|\pi)).\end{aligned} \tag{2.13}$$

Proof. Suppose $C(x,\lambda)$ and $S(x,\lambda)$ are the solutions of (1.1) with the initial conditions

$$C(0,\lambda)=1,\ C'(0,\lambda)=0 \text{ and } S(0,\lambda)=0,\ S'(0,\lambda)=1,$$

and the jump conditions (1.3). Clearly

$$\varphi(x,\lambda)=(\lambda-h_2)C(x,\lambda)+(h_3-\lambda h_1)S(x,\lambda).$$

The arguments for obtaining the asymptotic formulas of $S(x,\lambda)$ and $C(x,\lambda)$ are similar to that of [38]. Note that by changing x to $\pi - x$ one can obtain the asymptotic form of $\psi(x,\lambda)$ and $\psi'(x,\lambda)$. □

By applying the similar calculations of [1, 38], we find that

$$\rho_n = n - 2 + \frac{\theta_n}{n-2} + \frac{\kappa_n}{n}, \tag{2.14}$$

where

$$\kappa_n = o(1), \quad \theta_n = \frac{(-1)^{(n+1)}}{2}\left(\omega_1 + \omega_2 \cos 2d(n-2)\right),$$

and

$$\omega_1 = a\left(H_1 + h_1 - \frac{1}{2}\int_0^\pi q(t)\,dt\right) - \frac{b}{2}, \quad \omega_2 = -\frac{b}{2}.$$

3. Main results

In this section the uniqueness theorem for Eqs. (1.1)–(1.3) is given. We need some lemma and technical notation to prove our main result. The boundary value problem $L = L(q; h_i; H_i; d)$ is defined with the operator $A : \mathcal{H} \to \mathcal{H}$. We now consider boundary value problems $\tilde{L} := L(\tilde{q}; h_i; H_i; d)$, $L_1 := L(q; h_i; \mathfrak{H}_i; d)$, and $\tilde{L}_1 := L(\tilde{q}; h_i; \mathfrak{H}_i; d)$, for $i = 1, 2, 3$, by the same approach with operators $\tilde{A}$, A_1, and $\tilde{A}_1$ respectively, where $\mathfrak{H}_1 \neq H_1$. Suppose that $\theta(x,\lambda)$ is the solution of (1.1) satisfying in the initial conditions $\theta(\pi,\lambda) = \mathfrak{H}_2 - \lambda$, $\theta'(\pi,\lambda) = \lambda\mathfrak{H}_1 - \mathfrak{H}_3$ and the jump conditions (1.3). Define $\phi_j(\lambda) := W(\varphi_j(x,\lambda), \theta_j(x,\lambda))$, and $\tilde{\phi}_j(\lambda) := W(\tilde{\varphi}_j(x,\lambda), \tilde{\theta}_j(x,\lambda))$, for $j = 1, 2$.

Lemma 3.1. *If $L(q; h_i; \mathfrak{H}_i; d)$ and $L(\tilde{q}; h_i; \mathfrak{H}_i; d)$, $(i = 1, 2, 3)$, have the same eigenvalues, then $\phi_j(\lambda) = \tilde{\phi}_j(\lambda)$, for $j = 1, 2$.*

Proof. From [8] it follows that ϕ and $\tilde{\phi}$ are entire functions of order $\frac{1}{2}$, and consequently, using Hadamard's factorization theorem [23] are determined up to a multiplicative constant by their zeros. Hence there is a constant k such that $k = \frac{\phi_j(\lambda)}{\tilde{\phi}_j(\lambda)}$. Using the asymptotic form of $\phi_j(\lambda)$ and $\tilde{\phi}_j(\lambda)$ as a similar form of (2.13) with H_i replaced by $\mathfrak{H}_i$, we obtain $k = 1 + O(\frac{1}{\rho})$. Letting $\rho \to \infty$, we obtain $k = 1$ and so $\phi_j(\lambda) = \tilde{\phi}_j(\lambda)$. □

If $\psi_n(x) := \psi(x,\lambda_n)$ is another eigenfunction of L satisfying in the initial conditions (2.6), then $\varphi_n(x)$ and $\psi_n(x)$ are linearly dependent for $n \in \mathbb{N}$. So, we have

$$\psi_n(x) = k_n \varphi_n(x), \quad x \in [0, d) \cup (d, \pi], \tag{3.1}$$

where k_n is a real number. Define $\tilde{\varphi}_n(x)$, $\tilde{\psi}_n(x)$ and $\tilde{k}_n$ in a similar manner. From this on, we assume that $\Lambda_0 \subseteq \mathbb{N}$ is a finite set and $\Lambda = \mathbb{N} \backslash \Lambda_0$.

Lemma 3.2. *If L_1 and $\tilde{L}_1$ have the same eigenvalues and, as well as, $\lambda_n = \tilde{\lambda}_n$ for all $n \in \Lambda$, where λ_n and $\tilde{\lambda}_n$ are the eigenvalues of L and $\tilde{L}$, respectively, then $k_n = \tilde{k}_n$ for all $n \in \Lambda$.*

Proof. Define $\delta_j(\lambda) := W(\psi_j(x,\lambda), \theta_j(x,\lambda))$. It is easy to see that $\delta_j(\lambda)$ is independent of x. From definition of ϕ, θ and ψ it follows that

$$\begin{cases} W(\varphi_{jn}(x), \psi_{jn}(x)) = 0, \\ W(\varphi_{jn}(x), \theta_{jn}(x)) = \phi_j(\lambda_n), \end{cases} \tag{3.2}$$

for $j = 1, 2$. The above linear system has a unique solution

$$\varphi_{jn}(x) = \frac{\psi_{jn}(x)\phi_j(\lambda_n)}{\delta_j(\lambda_n)}, \quad \varphi'_{jn}(x) = \frac{\psi'_{jn}(x)\phi_j(\lambda_n)}{\delta_j(\lambda_n)}. \tag{3.3}$$

Similarly we obtain

$$\tilde{\varphi}_{jn}(x) = \frac{\tilde{\psi}_{jn}(x)\tilde{\phi}_j(\tilde{\lambda}_n)}{\tilde{\delta}_j(\tilde{\lambda}_n)}, \quad \tilde{\varphi}'_{jn}(x) = \frac{\tilde{\psi}'_{jn}(x)\tilde{\phi}_j(\tilde{\lambda}_n)}{\tilde{\delta}_j(\tilde{\lambda}_n)}. \tag{3.4}$$

From $\lambda_n = \tilde{\lambda}_n$ for all $n \in \Lambda$ and Lemma 3.1, we have $\phi_j \equiv \tilde{\phi}_j$. From definition of $\delta_j(\lambda)$ it follows that

$$\delta_2(\lambda_n) = \tilde{\delta}_2(\lambda_n)|_{x=\pi} = \lambda_n^2(H_1 - \mathfrak{H}_1) + \lambda_n(H_2\mathfrak{H}_1 - \mathfrak{H}_2H_1 + \mathfrak{H}_3 - H_3) + H_3\mathfrak{H}_2 - H_2\mathfrak{H}_3.$$

Thus

$$k_n = \tilde{k}_n = \frac{\lambda_n^2(H_1 - \mathfrak{H}_1) + \lambda_n(H_2\mathfrak{H}_1 - \mathfrak{H}_2H_1 + \mathfrak{H}_3 - H_3) + H_3\mathfrak{H}_2 - H_2\mathfrak{H}_3}{\phi_2(\lambda_n)}$$

for all $n \in \Lambda$. □

Assume that λ is not in the spectrum of (1.1)–(1.3) and let

$$S_\lambda := (A - \lambda I)^{-1}|_D.$$

Replace A by $\tilde{A}$ and define $\tilde{S}_\lambda$ analogously.

We consider the following spaces

$$K := D(A) \ominus \{\Phi_m : m \in \Lambda_0\}, \tag{3.5}$$

$$\tilde{K} := D(\tilde{A}) \ominus \{\tilde{\Phi}_m : m \in \Lambda_0\}. \tag{3.6}$$

Define the transformation operator $T : K \to \tilde{K}$ by

$$T\Phi_n = \tilde{\Phi}_n, \tag{3.7}$$

where $\Phi_n = \begin{pmatrix} \varphi_n(x) \\ R_1(\varphi_n) \\ R_2(\varphi_n) \end{pmatrix}$ and $\tilde{\Phi}_n = \begin{pmatrix} \tilde{\varphi}_n(x) \\ R_1(\tilde{\varphi}_n) \\ R_2(\tilde{\varphi}_n) \end{pmatrix}$ for $n \in \Lambda$. By using the asymptotic form of solutions (2.11) and (2.12), it is easy to verify that T is a bounded operator. From (2.4) we have

$$(\lambda I - A)\Phi_n = (\lambda - \lambda_n)\Phi_n,$$

thus we obtain

$$\frac{\Phi_n}{(\lambda - \lambda_n)} = -S_\lambda \Phi_n.$$

A similar relation is obviously valid for $\tilde{\Phi}_n$.

Lemma 3.3. *The relation $\tilde{S}_\lambda T = T S_\lambda$ holds for $\lambda \neq \lambda_n, \tilde{\lambda}_n$ and $n \in \mathbb{N}$.*

Proof. Let $F \in K$, then we can expand F in terms of the set Φ_n

$$F(x) = \begin{pmatrix} f(x) \\ R_1(f) \\ R_2(f) \end{pmatrix} = \sum_{\Lambda} f_n \Phi_n(x), \tag{3.8}$$

for $n \in \Lambda$, where $f_n = \frac{\langle F, \Phi_n \rangle_{\mathcal{H}}}{\langle \Phi_n, \Phi_n \rangle_{\mathcal{H}}}$. Let λ be in complex plane which is not an eigenvalue of $A(q; h_i; H_i; d)$, then the operator S_λ exists and can be written as

$$-S_\lambda F(x) = \sum_{\Lambda} \frac{f_n}{\lambda - \lambda_n} \Phi_n(x). \tag{3.9}$$

If we apply T to the above relation, we obtain

$$-T S_\lambda F(x) = \sum_{\Lambda} \frac{f_n}{\lambda - \lambda_n} \tilde{\Phi}_n(x).$$

If we apply $\tilde{S}_\lambda$ and T to (3.8) respectively, we obtain

$$-\tilde{S}_\lambda T F(x) = \sum_{\Lambda} \frac{f_n}{\lambda - \lambda_n} \tilde{\Phi}_n(x).$$

Then we get

$$\tilde{S}_\lambda T = T S_\lambda.$$

$\square$

In a general case when the operator L have spectral parameter dependent boundary conditions and discontinuous conditions, we generalize the well-known result of Hochstadt [13]. We construct the Green's function for the operator A by using its solutions $\varphi(x,\lambda)$ and $\psi(x,\lambda)$. By applying the Green's function we now prove our main theorem.

Theorem 3.4. *If $A(q;h_i;\mathfrak{H}_i;d)$ and $A(\tilde{q};h_i;\mathfrak{H}_i;d)$ have the same spectrum and $\lambda_n = \tilde{\lambda}_n$ for all $n\in\Lambda$, then*

$$q(x)-\tilde{q}(x) = \begin{cases} \sum_{\Lambda_0}(\tilde{y}_{1n}\varphi_{1n})'(x), & x<d, \\ \\ \sum_{\Lambda_0}(\tilde{y}_{2n}\varphi_{2n})'(x), & x>d, \end{cases} \tag{3.10}$$

a.e. on $[0,d)\cup(d,\pi]$, where $\tilde{y}_{in}$ and φ_{in} for $i=1,2$ are suitable solutions of $\tilde{l}y=\lambda_n y$ and $ly=\lambda_n y$, respectively.

Proof. By using the same techniques of [1] for $-S_\lambda\Phi_n = \mathcal{G}_n$, where $\mathcal{G}_n(x) = (g_n(x), R_1(g_n), R_2(g_n))^T \in \mathcal{H}$, by simple calculation we can show that the relation

$$g_n''(x) + (\lambda - q(x))g_n(x) = \varphi_n(x), \quad x\in(0,d)\cup(d,\pi), \tag{3.11}$$

$$\begin{aligned} &\lambda(g_n'(0)+h_1g_n(0)) - h_2g_n'(0) - h_3g_n(0) = 0, \\ &\lambda(g_n'(\pi)+H_1g_n(\pi)) - H_2g_n'(\pi) - H_3g_n(\pi) = 0, \end{aligned} \tag{3.12}$$

and

$$U_1(g_n)=0, \qquad U_2(g_n)=0 \tag{3.13}$$

are satisfied. The equation (3.11) with (3.12) and (3.13) has the unique solution (i.e. $g_n(x)$), which can be represented as

$$g_n(x) = \begin{cases} \frac{\psi_1(x,\lambda)}{\Delta_1(\lambda)}\int_0^x \varphi_1(t,\lambda)\varphi_{1n}(t)dt + \frac{\varphi_1(x,\lambda)}{\Delta_1(\lambda)}\big(\int_x^d \psi_1(t,\lambda)\varphi_{1n}(t)dt \\ \qquad\qquad + \frac{1}{a^2}\int_d^\pi \psi_2(t,\lambda)\varphi_{2n}(t)dt\big), & 0<x<d, \\ \\ \frac{\psi_2(x,\lambda)}{\Delta_2(\lambda)}\Big(a^2\int_0^d \varphi_1(t,\lambda)\varphi_{1n}(t)dt + \int_d^x \varphi_2(t,\lambda)\varphi_{2n}(t)dt\Big) \\ \qquad\qquad + \frac{\varphi_2(t,\lambda)}{\Delta_2(\lambda)}\int_x^\pi \psi_2(t,\lambda)\varphi_{2n}(t)dt, & d<x<\pi. \end{cases} \tag{3.14}$$

By considering

$$G(x,t,\lambda) = \begin{cases} |a|\frac{\psi(x,\lambda)\varphi(t,\lambda)}{\Delta(\lambda)}, & 0\le t\le x\le\pi, \\ |a|\frac{\varphi(x,\lambda)\psi(t,\lambda)}{\Delta(\lambda)}, & 0\le x\le t\le\pi, \end{cases} \tag{3.15}$$

where $x \neq d$ and $t \neq d$ the formula (3.14) reduces to

$$\mathcal{G}_n(x) = \begin{pmatrix} g_n(x) \\ R_1(g_n) \\ R_2(g_n) \end{pmatrix} = \begin{pmatrix} |a| \int_0^d G(x,t,\lambda)\varphi_{1n}(t)dt + \frac{1}{|a|}\int_d^\pi G(x,t,\lambda)\varphi_{2n}(t)dt \\ \frac{R_1(\varphi_n)}{\lambda-\lambda_n} \\ \frac{R_2(\varphi_n)}{\lambda-\lambda_n} \end{pmatrix} \tag{3.16}$$

and the function $G(x,t,\lambda)$ is as defined in (3.15). Using the asymptotic form of $\varphi(x,\lambda)$, $\psi(x,\lambda)$, $\Delta(\lambda)$ for sufficiently large ρ and $\rho \neq \rho_n$, we deduce that the Green's function $G(x,t,\lambda)$ is bounded. $G(x,t,\lambda)$ is a meromorphic function with the eigenvalues λ_k as its poles [1]. Let C_n be a sequence of circles about the origin intersecting the positive λ-axis between λ_n and λ_{n+1}. We have

$$\lim_{n\to\infty} \int_{C_n} \frac{G(x,t,\mu)}{\lambda-\mu} d\mu = 0, \qquad \lambda \in \text{int } C_n. \tag{3.17}$$

From residue integration, it follows that

$$\frac{1}{2\pi i} \int_{C_n} \frac{G(x,t,\mu)}{\lambda-\mu} d\mu = -G(x,t,\lambda) + \sum_{i=0}^{n} \frac{\varphi_i(x<)\psi_i(x>)}{\dot{\Delta}(\lambda_i)(\lambda-\lambda_i)}, \tag{3.18}$$

where $\dot{\Delta}(\lambda_i) = \frac{d}{d\lambda}\Delta(\lambda)|_{\lambda=\lambda_i}$. From (3.17), (3.18) and the Mittag-Leffler expansion for $G(x,t,\lambda)$ we obtain

$$G(x,t,\lambda) = \sum_{i=0}^{\infty} \frac{\varphi_i(x<)\psi_i(x>)}{\dot{\Delta}(\lambda_i)(\lambda-\lambda_i)}, \tag{3.19}$$

where for simplicity $x <:= \min\{x,t\}$ and $x >:= \max\{x,t\}$ and $\varphi_i(x<)$, $\psi_i(x>)$ are eigenfunctions corresponding to the eigenvalues λ_i, therefore for $(f(x), R_1(f), R_2(f))^T \in K$, from (3.5), (3.15), (3.16), and Lemma 3.2 we have

$$
\begin{aligned}
&(y(x), R_1(y), R_2(y))^T \\
&= S_\lambda\left(f(x), R_1(f), R_2(f)\right)^T \\
&= S_\lambda F(x) \\
&= \begin{pmatrix}
\begin{cases}
a^2 \frac{\psi_1(x)}{\Delta_2(\lambda)} \int_0^x \varphi_1(t) f(t)dt + \frac{\varphi_1(x)}{\Delta_2(\lambda)}\left(a^2 \int_x^d \psi_1(t) f(t)dt + \int_d^\pi \psi_2(t) f(t)dt\right), \\
\qquad\qquad 0 \le x < d, \\
\frac{\psi_2(x)}{\Delta_2(\lambda)}\left(a^2 \int_0^d \varphi_1(t) f(t)dt + \int_d^x \varphi_2(t) f(t)dt\right) + \frac{\varphi_2(x)}{\Delta_2(\lambda)} \int_x^\pi \psi_2(t) f(t)dt, \\
\qquad\qquad d < x \le \pi,
\end{cases} \\
R_1(y) \\
R_2(y)
\end{pmatrix}, \\
&= \begin{pmatrix}
\begin{cases}
\sum_\Lambda \frac{a^2 \psi_{1n}(x) \int_0^x \varphi_{1n}(t) f(t)dt + \varphi_{1n}(x)\left(a^2 \int_x^d \psi_{1n}(t) f(t)dt + \int_d^\pi \psi_{2n}(t) f(t)dt\right)}{\dot{\Delta}(\lambda_n)(\lambda-\lambda_n)}, \\
\qquad\qquad 0 \le x < d, \\
\sum_\Lambda \frac{\psi_{2n}(x)(a^2 \int_0^d \varphi_{1n}(t) f(t)dt + \int_d^x \varphi_{2n}(t) f(t)dt) + \varphi_{2n}(x) \int_x^\pi \psi_{2n}(t) f(t)dt}{\dot{\Delta}(\lambda_n)(\lambda-\lambda_n)}, \\
\qquad\qquad d < x \le \pi,
\end{cases} \\
\sum_\Lambda \frac{f_n R_1(\varphi_n)}{\lambda-\lambda_n} \\
\sum_\Lambda \frac{f_n R_2(\varphi_n)}{\lambda-\lambda_n}
\end{pmatrix}, \\
&= \begin{pmatrix}
\begin{cases}
\sum_\Lambda \frac{k_n \varphi_{1n}(x)\left(a^2 \int_0^d \varphi_{1n}(t) f(t)dt + \int_d^\pi \varphi_{2n}(t) f(t)dt\right)}{\dot{\Delta}(\lambda_n)(\lambda-\lambda_n)}, & 0 \le x < d, \\
\sum_\Lambda \frac{k_n \varphi_{2n}(x)\left(a^2 \int_0^d \varphi_{1n}(t) f(t)dt + \int_d^\pi \varphi_{2n}(t) f(t)dt\right)}{\dot{\Delta}(\lambda_n)(\lambda-\lambda_n)}, & d < x \le \pi,
\end{cases} \\
\sum_\Lambda \frac{f_n R_1(\varphi_n)}{\lambda-\lambda_n} \\
\sum_\Lambda \frac{f_n R_2(\varphi_n)}{\lambda-\lambda_n}
\end{pmatrix}
\end{aligned}
\tag{3.20}
$$

By applying T to both sides of (3.20), we see that

$$
TS_\lambda F(x) = \begin{pmatrix}
\begin{cases}
\sum_\Lambda \frac{k_n \tilde{\varphi}_{1n}(x)\left(a^2 \int_0^d \varphi_{1n}(t) f(t)dt + \int_d^\pi \varphi_{2n}(t) f(t)dt\right)}{\dot{\Delta}(\lambda_n)(\lambda-\lambda_n)}, & 0 \le x < d, \\
\sum_\Lambda \frac{k_n \tilde{\varphi}_{2n}(x)\left(a^2 \int_0^d \varphi_{1n}(t) f(t)dt + \int_d^\pi \varphi_{2n}(t) f(t)dt\right)}{\dot{\Delta}(\lambda_n)(\lambda-\lambda_n)}, & d < x \le \pi,
\end{cases} \\
\sum_\Lambda \frac{f_n R_1(\tilde{\varphi}_n)}{\lambda-\lambda_n} \\
\sum_\Lambda \frac{f_n R_2(\tilde{\varphi}_n)}{\lambda-\lambda_n}
\end{pmatrix}. \tag{3.21}
$$

Define

$$
U(x) := \begin{pmatrix}
\begin{cases}
\frac{a^2 \tilde{\psi}_1(x) \int_0^x \varphi_1(t) f(t)dt + \tilde{\varphi}_1(x)\left(a^2 \int_x^d \psi_1(t) f(t)dt + \int_d^\pi \psi_2(t) f(t)dt\right)}{\Delta(\lambda)}, \\
\qquad\qquad 0 \le x < d, \\
\frac{\tilde{\psi}_2(x)\left(a^2 \int_0^d \varphi_1(t) f(t)dt + \int_d^x \varphi_2(t) f(t)dt\right) + \tilde{\varphi}_2(x) \int_x^\pi \psi_2(t) f(t)dt}{\Delta(\lambda)}, \\
\qquad\qquad d < x \le \pi,
\end{cases} \\
R_1(y) \\
R_2(y)
\end{pmatrix}. \tag{3.22}
$$

By the Mittag-Leffler expansion for $U(x)$, we have

$$U(x)=\begin{pmatrix} \begin{array}{l} \sum_{\Lambda_0}\frac{a^2\tilde{w}_{1n}(x)\int_0^x\varphi_{1n}(y)f(y)dy+\tilde{z}_{1n}(x)\left(a^2\int_x^d\psi_{1n}(t)f(t)dt+\int_d^\pi\psi_{2n}(t)f(t)dt\right)}{\dot{\Delta}(\lambda_n)(\lambda-\lambda_n)}\\ +\sum_{\Lambda}\frac{a^2\tilde{\psi}_{1n}(x)\int_0^x\varphi_{1n}(t)f(t)dt+\tilde{\varphi}_{1n}(x)\left(a^2\int_x^d\psi_{1n}(t)f(t)dt+\int_d^\pi\psi_{2n}(t)f(t)dt\right)}{\dot{\Delta}(\lambda_n)(\lambda-\lambda_n)},\\ \qquad\qquad 0\le x<d,\\ \sum_{\Lambda_0}\frac{\tilde{w}_{2n}(x)\left(a^2\int_0^d\varphi_{1n}(t)f(t)dt+\int_x^d\varphi_{2n}(t)f(t)dt\right)+\tilde{z}_{2n}(x)\int_x^\pi\psi_{2n}(t)f(t)dt}{\dot{\Delta}(\lambda_n)(\lambda-\lambda_n)}\\ +\sum_{\Lambda}\frac{\tilde{\psi}_{2n}(x)\left(a^2\int_0^d\varphi_{1n}(t)f(t)dt+\int_x^d\varphi_{2n}(t)f(t)dt\right)+\tilde{\varphi}_{2n}(x)\int_x^\pi\psi_{2n}(t)f(t)dt}{\dot{\Delta}(\lambda_n)(\lambda-\lambda_n)},\\ \qquad\qquad d<x\le\pi, \end{array}\\ \\ \sum_{n\in\mathbb{N}}\frac{f_nR_1(\tilde{\varphi}_n)}{\lambda-\lambda_n}\\ \sum_{n\in\mathbb{N}}\frac{f_nR_2(\tilde{\varphi}_n)}{\lambda-\lambda_n} \end{pmatrix}.$$

The second term of the above expression is $TS_\lambda F$, as given in (3.21), in the first term, $\tilde{w}_n(x)$ represents $\tilde{\psi}(x)$ and $\tilde{z}_n(x)$ represents $\tilde{\varphi}(x)$ evaluated at λ_n. Hence

$$\tilde{S}_\lambda TF(x)=U(x)-\begin{pmatrix} \begin{cases} \sum_{\Lambda_0}\frac{a^2\tilde{w}_{1n}(x)\int_0^x\varphi_{1n}(t)f(t)dt+\tilde{z}_{1n}(x)\left(a^2\int_x^d\psi_{1n}(y)f(y)dy+\int_d^\pi\psi_{2n}(t)f(t)dt\right)}{\dot{\Delta}(\lambda_n)(\lambda-\lambda_n)}, & 0\le x<d,\\ \sum_{\Lambda_0}\frac{\tilde{w}_{2n}(x)\left(a^2\int_0^d\varphi_{1n}(t)f(t)dt+\int_x^d\varphi_{2n}(t)f(t)dt\right)+\tilde{z}_{2n}(x)\int_x^\pi\psi_{2n}(t)f(t)dt}{\dot{\Delta}(\lambda_n)(\lambda-\lambda_n)}, & d<x\le\pi, \end{cases}\\ \\ \sum_{\Lambda_0}\frac{f_nR_1(\tilde{\varphi}_n)}{\lambda-\lambda_n}\\ \sum_{\Lambda_0}\frac{f_nR_2(\tilde{\varphi}_n)}{\lambda-\lambda_n} \end{pmatrix}. \tag{3.23}$$

The right and left hand side of (3.23) is in the domain $\tilde{S}_\lambda$. Therefore, both sides of (3.23) are continuous. By using (3.20) and differentiation of the right-hand side of (3.23), for $0\le x<d$ we obtain

$$\Bigg[\frac{a^2\tilde{\psi}_1'(x)\int_0^x\varphi_1(t)f(t)dt+\tilde{\varphi}_1'(x)\left(a^2\int_x^d\psi_1(t)f(t)dt+\int_d^\pi\psi_2(t)f(t)dt\right)}{\Delta(\lambda)}$$

$$-\sum_{\Lambda_0}\frac{a^2\tilde{w}_{1n}'(x)\int_0^x\varphi_{1n}(t)f(t)dt+\tilde{z}_{1n}'(x)\left(a^2\int_x^d\psi_{1n}(t)f(t)dt+\int_d^\pi\psi_{2n}(t)f(t)dt\right)}{\dot{\Delta}(\lambda_n)(\lambda-\lambda_n)}\Bigg]$$

$$+\left[\frac{\tilde{\psi}_1(x)\varphi_1(x)-\tilde{\varphi}_1(x)\psi_1(x)}{\Delta(\lambda)}-\sum_{\Lambda_0}\frac{\tilde{w}_{1n}(x)\varphi_{1n}(x)+\tilde{z}_{1n}(x)\psi_{1n}(x)}{\dot{\Delta}(\lambda_n)(\lambda-\lambda_n)}\right]f(x).$$

An inspection of the term in the second set of braces shows that it vanishes identically. To do that, one merely computes the residue at each λ_n and observes that it becomes zero. One can differentiate the expression in the braces in the last expression and then from (3.23) we obtain

$$Tf(x) = \left[\frac{\tilde{\psi}_1(x)\varphi_1(x) - \tilde{\varphi}_1(x)\psi_1(x)}{\Delta(\lambda)} - \sum_{\Lambda_0} \frac{\tilde{w}_{1n}(x)\varphi_{1n}(x) + \tilde{z}_{1n}(x)\psi_{1n}(x)}{\dot{\Delta}(\lambda_n)(\lambda - \lambda_n)} \right] f(x)$$
$$- \sum_{\Lambda_0} \frac{a^2 \tilde{w}_{1n}(x) \int_0^x \varphi_{1n}(t)f(t)dt + \tilde{z}_{1n}(x)\left(a^2 \int_x^d \psi_{1n}(t)f(t)dt + \int_d^\pi \psi_{2n}(t)f(t)dt\right)}{\dot{\Delta}(\lambda_n)}. \tag{3.24}$$

The operator T is independent of λ. To compute the value of the expression in the braces in (3.24) we let $\lambda \to \infty$. Using the asymptotic formulas, we see that the term in the braces reduces to unity. To simplify the second term in (3.24) we recall that $\psi_{1n} = k_n \varphi_{1n}$, $\psi_{2n} = k_n \varphi_{2n}$ and from (2.1),

$$|a| \int_0^d \psi_{1n}(t)f(t)dt + \frac{1}{|a|} \int_d^\pi \psi_{2n}(t)f(t)dt$$
$$+ \frac{|a|}{r_1} R_1(\psi_{1n})R_1(f) + \frac{1}{r_2|a|} R_2(\psi_{2n})R_2(f) = 0.$$

Then from (3.8), for $0 \le x < d$ we get

$$Tf(x) = f(x) - \frac{1}{2} \sum_{\Lambda_0} \tilde{y}_{1n}(x) \int_0^x \varphi_{1n}(t)f(t)dt$$
$$+ \sum_{\Lambda_0} \frac{f_n k_n \tilde{z}_{1n}(x)}{\dot{\Delta}(\lambda_n)} \left(\frac{|a|}{r_1} R_1(\psi_{1n})R_1(\varphi_{1n}) + \frac{1}{r_2|a|} R_2(\psi_{2n})R_2(\varphi_{2n}) \right), \tag{3.25}$$

where

$$\frac{1}{2}\tilde{y}_{1n}(x) = a^2 \frac{\tilde{w}_{1n}(x) - k_n \tilde{z}_{1n}(x)}{\dot{\Delta}(\lambda_n)}.$$

By applying the similar computation for $d < x \le \pi$ we obtain

$$Tf(x) = f(x) + \frac{1}{2} \sum_{\Lambda_0} \tilde{y}_{2n}(x) \int_x^\pi \varphi_{2n}(t)f(t)dt$$
$$+ \sum_{\Lambda_0} \frac{f_n \tilde{w}_{2n}(x)}{\dot{\Delta}(\lambda_n)} \left(\frac{|a|}{r_1} R_1(\psi_{1n})R_1(\varphi_{1n}) + \frac{1}{r_2|a|} R_2(\psi_{2n})R_2(\varphi_{2n}) \right),$$

where

$$\frac{1}{2}\tilde{y}_{2n}(x) = \frac{\tilde{w}_{2n}(x) - k_n \tilde{z}_{2n}(x)}{\dot{\Delta}(\lambda_n)}.$$

Now, from Lemma 3.3, we conclude that

$$\tilde{A}TF = TAF. \tag{3.26}$$

Suppose that $F = \Phi_n$ $(n \in \Lambda)$ then we get $f_m = \frac{\langle \Phi_n, \Phi_m \rangle_{\mathcal{H}}}{\langle \Phi_m, \Phi_m \rangle_{\mathcal{H}}} = 0$, for $m \in \Lambda_0$. For left and right side of (3.26) we get

$$\begin{aligned}
\tilde{A}T\Phi_n =& \tilde{A}\begin{pmatrix} \begin{cases} \varphi_{1n} - \frac{1}{2}\sum_{\Lambda_0} \tilde{y}_{1m} \int_0^x \varphi_{1m}(t)\varphi_{1n}(t)dt, & 0 \leq x < d, \\ \varphi_{2n} + \frac{1}{2}\sum_{\Lambda_0} \tilde{y}_{2m}(x) \int_x^\pi \varphi_{2m}(t)\varphi_{2n}(t)dt, & d < x \leq \pi, \end{cases} \\ \\ R_1(\tilde{\varphi}_n) \\ R_2(\tilde{\varphi}_n) \end{pmatrix}, \\
=& \begin{pmatrix} \begin{cases} -\varphi''_{1n} + \tilde{q}\varphi_{1n} - \frac{1}{2}\sum_{\Lambda_0} \tilde{\ell}\left(\tilde{y}_{1m} \int_0^x \varphi_{1m}(t)\varphi_{1n}(t)dt\right), & 0 \leq x < d, \\ -\varphi''_{2n} + \tilde{q}\varphi_{2n} + \frac{1}{2}\sum_{\Lambda_0} \tilde{\ell}\left(\tilde{y}_{2m}(x) \int_x^\pi \varphi_{2m}(t)\varphi_{2n}(t)dt\right), & d < x \leq \pi, \end{cases} \\ \\ R'_1(\tilde{\varphi}_n) \\ R'_2(\tilde{\varphi}_n) \end{pmatrix} \\
=& \begin{pmatrix} \begin{cases} -\varphi''_{1n} + \tilde{q}\varphi_{1n} - \frac{1}{2}\sum_{\Lambda_0} \tilde{\ell}\tilde{y}_{1m} \int_0^x \varphi_{1m}(t)\varphi_{1n}(t)dt \\ \quad + \frac{1}{2}\sum_{\Lambda_0} 2\tilde{y}'_{1m}(\varphi_{1m}\varphi_{1n}) + \tilde{y}_{1m}(\varphi_{1m}\varphi_{1n})', & 0 \leq x < d, \\ -\varphi''_{2n} + \tilde{q}\varphi_{2n} + \frac{1}{2}\sum_{\Lambda_0} \tilde{\ell}\tilde{y}_{2m} \int_x^\pi \varphi_{2m}(t)\varphi_{2n}(t)dt \\ \quad + \frac{1}{2}\sum_{\Lambda_0} (-2)\tilde{y}'_{2m}(\varphi_{2m}\varphi_{2n}) + \tilde{y}_{2m}(\varphi_{2m}\varphi_{2n})', & d < x \leq \pi, \end{cases} \\ \\ R'_1(\tilde{\varphi}_n) \\ R'_2(\tilde{\varphi}_n) \end{pmatrix}
\end{aligned} \tag{3.27}$$

and

$$\begin{aligned}
TA\Phi_{1n} =& \begin{pmatrix} \begin{cases} -\varphi''_{1n} + q\varphi_{1n} - \frac{1}{2}\sum_{\Lambda_0} \tilde{y}_{1m} \int_0^x \varphi_{1m}\ell\varphi_{1n}, & 0 \leq x < d, \\ -\varphi''_{2n} + q\varphi_{2n} + \frac{1}{2}\sum_{\Lambda_0} \tilde{y}_{2m} \int_x^\pi \varphi_{2m}\ell\varphi_{2n}, & d < x \leq \pi, \end{cases} \\ \\ R'_1(\tilde{\varphi}_n) \\ R'_2(\tilde{\varphi}_n) \end{pmatrix} \\
=& \begin{pmatrix} \begin{cases} -\varphi''_{1n} + q\varphi_{1n} - \frac{1}{2}\sum_{\Lambda_0} \tilde{y}_{1m} \int_0^x \varphi_{1n}\ell\varphi_{1m} \\ \quad - \frac{1}{2}\sum_{\Lambda_0} \tilde{y}_{1m}(\varphi_{1n}\varphi'_{1m} - \varphi_{1m}\varphi'_{1n}), & 0 \leq x < d, \\ -\varphi''_{2n} + q\varphi_{2n} + \frac{1}{2}\sum_{\Lambda_0} \tilde{y}_{2m} \int_x^\pi \varphi_{2n}\ell\varphi_{2m} \\ \quad + \frac{1}{2}\sum_{\Lambda_0} \tilde{y}_{2m}(\varphi_{2n}\varphi'_{2m} - \varphi_{2m}\varphi'_{2n}), & d < x \leq \pi, \end{cases} \\ \\ R'_1(\tilde{\varphi}_n) \\ R'_2(\tilde{\varphi}_n) \end{pmatrix}.
\end{aligned} \tag{3.28}$$

Note that

$$\begin{aligned}
\sum_{\Lambda_0} \tilde{y}_{1m} \int_0^x \varphi_{1n}\ell\varphi_{1m} =& \sum_{\Lambda_0} \tilde{y}_{1m} \int_0^x \lambda_m \varphi_{1m}\varphi_{1n} \\
=& \sum_{\Lambda_0} \lambda_m \tilde{y}_{1m} \int_0^x \varphi_{1m}\varphi_{1n} \\
=& \sum_{\Lambda_0} \tilde{\ell}\tilde{y}_{1m} \int_0^x \varphi_{1m}\varphi_{1n}
\end{aligned}$$

and

$$\sum_{\Lambda_0} \tilde{y}_{2m} \int_x^\pi \varphi_{2n} \ell \varphi_{2m} = \sum_{\Lambda_0} \tilde{\ell} \tilde{y}_{2m} \int_x^\pi \varphi_{2m} \varphi_{2n}.$$

Using (3.26) we find that

$$q(x) - \tilde{q}(x) = \begin{cases} \sum_{\Lambda_0} (\tilde{y}_{1m}\varphi_{1m})', & 0 \le x < d, \\ \sum_{\Lambda_0} (\tilde{y}_{2m}\varphi_{2m})', & d < x \le \pi. \end{cases}$$

□

Corollary 3.5. *If Λ_0 is empty, then T is a unitary operator and $A = \tilde{A}$. Hence $q = \tilde{q}$ in $L^2(0, \pi)$.*

4. Conclusion

In this paper, the inverse Sturm–Liouville problems with a transmission and parameter dependent boundary conditions was studied. For this purpose, a new Hilbert space by defining a new inner product for obtaining a self–adjoint operator was defined. So, the asymptotic form of solutions, eigenvalues and eigenfunctions of this problem was obtained. Finally, we formulated the Hochestadt's result based on transformation operator for inverse Sturm–Liouville problems.

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References

[1] Z. Akdoğan, M. Demirci and O. Sh. Mukhtarov, *Green function of discontinuous boundary–value problem with transmission conditions*, Math. Meth. Appl. Sci., 30 (2007) 1719–1738.

[2] S. Albeverio, F. Gesztesy, R. Høegh-Krohn, and H. Holden, *Solvable Models in Quantum Mechanics*, 2nd ed., AMS Chelsea Publishing, Providence, RI, 2005.

[3] V. A. Ambartsumyan, *Über eine frage der eigenwerttheorie*, Z. Phys. 53 (1929) 690–695.

[4] R. Kh. Amirov, *On Sturm–Liouville operators with discontinuity conditions inside an interval*, J. Math. Anal. Appl. 317 (2006) 163–176.

[5] J. Ben Amara and A. A. Shkalikov, *A Sturm-Liouville Problem With Physical and Spectral Parameters in Boundary Conditions*, Mathematical Notes, Vol. 66, No. 2, 1999.

[6] P. A. Binding, P. J. Browne, and B. A. Watson, *Inverse spectral problems for left–definite Sturm–Liouville equations with indefinite weight*, J. Math. Anal. Appl. 271 (2002) 383–408.

[7] G. Borg, *Eine umkehrung der Sturm–Liouvilleschen eigenwertaufgabe*, Acta Math. 78 (1945) 1–96.

[8] E. A. Coddington and N. Levinson, *Theory of Ordinary Differential Equations*, Krieger, 1984.

[9] J. Eckhardt and G. Teschl, *Uniqueness results for Schrödinger operators on the line with purely discrete spectra*, arXiv:1110.2453.

[10] G. Freiling and V. A. Yurko, *Inverse Sturm–Liouville problems and their applications*, NOVA Science Publishers, New York, 2001.

[11] G. Freiling and V. A. Yurko, *Inverse problems for Sturm-Liouville equations with boundary conditions polynomially dependent on the spectral parameter*, Inverse Problems 26 (2010) 055003.

[12] O. Hald, *Discontinuous inverse eigenvalue problem*, Commun. Pure. Appl. Math. 37 (1984) 539–577.

[13] H. Hochstadt, *The inverse Sturm–Liouville problem*, Comm. pure. Math, 26 (1973) 715–729.

[14] H. Hochstadt, *On inverse problems associated with Sturm–Liouville operators*, J. Differential Equations, 17 (1975) 220–235.

[15] H. Hochstadt and B. Lieberman, *An inverse Sturm–Liouville problem with mixed given data*, SIAM J. Appl. Math., 34(4) (1978), 676–680.

[16] A. Jodayree Akbarfam and Angelo B. Mingarelli, *Duality for an indefinite inverse Sturm–Liouville problem*, J. Math. Anal. Appl. 312 (2005) 435–463.

[17] B. Keskin and A. S. Ozkan, *Inverse spectral problems for dirac operator with eigenvalue dependent boundary and jump conditions*, Acta Math. Hungar. 130 (4) (2011) 309–320.

[18] M. Kobayashi, *A uniqueness proof for discontinuous inverse Sturm-Liouville problems with symmetric potentials*, Inverse Problems 5 (1989) 767–781.

[19] H. Koyunbakan, *A new inverse problem for the diffusion operator*, Applied Mathematics Letters, 19 (2006) 995–999.

[20] H. Koyunbakan and E. S. Panakhov, *Inverse problem for a singular differential operator*, Mathematical and Computer Modeling, 47 (2008) 178–185.

[21] H. Koyunbakan, *Inverse problem for a quadratic pencil of Sturm–Liouville operator*, J. Math. Anal. Appl. 378 (2011) 549–554.

[22] R. J. Krueger, *Inverse problems for nonabsorbing media with discontinuous material properties*, J. Math. Phys. 23 (1982) 396–404.

[23] B.J. Levin, *Distribution of zeros of entire functions*, AMS. Transl. Vol. 5, Providence, 1964.

[24] B. M. Levitan, *Inverse Sturm–Liouville Problems*, VNU Science Press, 1987.

[25] N. Levinson, *The inverse Sturm–Liouville problem*, Mat. Tiddskr. 3 (1949) 25–30.

[26] V. A. Marchenko, *Sturm–Liouville operators and their applications*, Naukova Dumka, Kiev, 1977, English transl.: Birkhäuser, 1986.

[27] J. R. McLaughlin, *Analytical methods for recovering coefficients in differential equations from spectral data*, SIAM Rev. 28 (1986) 53–72.

[28] O. Sh. Mukhtarov, M. Kadakal and F. S. Muhtarov, *On discontinuous Sturm-Liouville problems with transmission conditions*, J. Math. Kyoto Univ. (JMKYAZ), 44-4 (2004) 779–798.

[29] Yu Pingwang, *An interior inverse problem for Sturm–Liouville operator with eigenparameter dependent boundary conditions*, TAMKANG J. Math., Vo. 42, No. 3, (2011) 395–403.

[30] M. Shahriari, *Inverse Sturm-Liouville problem with eigenparameter dependent boundary and transmission conditions*, Azerbaijan Journal of Mathematics, Vo. 4, No 2 (2014) 16–30.

[31] M. Shahriari, A. J. Akbarfama, G. Teschl, *Uniqueness for Inverse Sturm–Liouville Problems with a Finite Number of Transmission Conditions*, J. Math. Anal. Appl., 395 (2012) 19-29.

[32] Chung-Tsun Shieh and V.A. Yurko, *Inverse nodal and inverse spectral problems for discontinuous boundary value problems*, J. Math. Anal. Appl. 347 (2008) 266–272.

[33] G. Teschl, *Mathematical Methods in Quantum Mechanics; With Applications to Schrödinger Operators*, Graduate Studies in Mathematics 99, Amer. Math. Soc., Providence, RI, 2009.

[34] C. Willis, *Inverse Sturm-Liouville problems with two discontinuities*, Inverse Problems I (1985) 263-289.

[35] Chuan-Fu Yang and Xiao-Ping Yang, *An interior inverse problem for the Sturm–Liouville operator with discontinuous conditions*, Applied Mathematics Letters, 22 (2009) 1315–1319.

[36] Chuan-Fu Yang, *Hochstadt-Lieberman theorem for Dirac operator with eigenparameter dependent boundary conditions*, Nonlinear Analysis, 74 (2011) 2475–2484.

[37] Chuan-Fu Yang, *Inverse nodal problems for the Sturm-Liouville operator with eigenparameter dependent boundary conditions*, Operator and matrices, Volume 6, Number 1 (2012), 6377.

[38] V. Yurko, *Integral transforms connected with discontinuous boundary value problems*, Int. Trans. Spec. Functions, 10 (2000) 141–164.