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Inverse Laplace transform method for multiple solutions of the fractional Sturm-Liouville problems

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Abstract In this paper, inverse Laplace transform method is applied to analytical solution of the fractional Sturm-Liouville problems. The method introduces a powerful tool for solving the eigenvalues of the fractional Sturm-Liouville problems. The results show that the simplicity and efficiency of this method.

Keywords. Laplace transform, Fractional Sturm-Liouville problems, Caputo's fractional derivative, Eigenvalue.

1991 Mathematics Subject Classification.

1. Introduction

Laplace transform has solved basic differential equations since the late eighteenth century [8]. Modern problems, however, often require an extension of Laplace method to more challenging settings. Laplace transform has been considered as a useful tool to solve integer-order or relatively simple fractional-order differential [6, 9]. Inverse Laplace transform is an important step in the application of Laplace transformation technique in solving differential equations. The inverse Laplace transformation can accomplished analytically according to its definition, or by using Laplace transform tables. In this paper, we applied The inverse Laplace transform method for solving fractional Sturm-Liouville problems. The aim of this paper is to present an efficient and reliable treatment of the inverse Laplace transform method for solving fractional Sturm-Liouville problems.
In this paper, we consider the following class of eigenvalue problems of the from

$$D^{\alpha}[p(x)y^{'}(x)] + \lambda q(x)y(x) = 0, \qquad x \in (0,1), \quad 0 < \alpha \leq 1,$$

subject to

$$ay(0) + by^{'}(0) = 0, \qquad cy(1) + dy^{'}(1) = 0,$$

where $a, b, c, d \in \mathbb{R}$ and $q(x), p(x) > 0$, $q(x)$ and $p(x)$ are smooth functions. Here D^α denotes the fractional differential operator of order α.
Several authors have considered the numerical computational of such points, for example Al-Mdallal [2] applied the Adomian decomposition method for solving fractional Sturm-Liouville problems.
Abbasbandy [1] applied the Homotopy analysis method for solving fractional Sturm-Liouville problems.
The paper is organized as follows: In section 2 we introduce some necessary definitions and mathematical preliminaries of fractional calculus used for this study. Two illustrative examples are documented in Section 3. The last Section includes our conclusion.

2. Preliminaries

In this section, we give some definitions, notations and properties of fractional calculus used in this work.

Definition 2.1 A function $f(x)$, $x > 0$, is said to be in the space C_μ, $\mu \in \mathbb{R}$, if there exists a real number $p(> \mu)$, such that $f(x) = x^p f_1(x)$, which $f_1(x) \in C[0, \infty)$, and it is said to be in space C_μ^m if $f^{(m)} \in C_\mu$.
Definition 2.2 The left sided Riemann-Liouville fractional integral operator of order α of a function $f \epsilon\ C_\mu$, $\mu \geq -1$, is defined as [7]

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \frac{f(t)}{(x-t)^{1-\alpha}} dt, \qquad \alpha > 0, \qquad x > 0,$$
$$J^0 f(x) = f(x).$$

The Riemann-Liouville integral operator has the following properties

$$i) \quad J^\alpha J^\beta f(x) = J^\beta J^\alpha f(x),$$
$$ii) \quad J^\alpha J^\beta f(x) = J^{\alpha+\beta} f(x),$$
$$iii) \quad J^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}.$$

where $f \in C_\mu$, $\mu \geq -1$, $\alpha, \beta \geq 0$ and $\gamma > -1$.
Definition 2.3 Let $f \in C_{-1}^m$, $m \in \mathbb{N} \cup \{0\}$ then the Caputo's fractional derivative of $f(x)$ is defined as [7]

$$D^\alpha f(x) = \begin{cases} [J^{m-\alpha} f^{(m)}(x)] & m-1 < \alpha < m, \qquad m \in \mathbb{N}, \\ \frac{d^m f(x)}{dx^m}, & \alpha = m. \end{cases}$$

Definition 2.4 The Laplace transform of original function $f(x)$ of a real variable x, for $x \geq 0$, is defined by the integral (if it exists)

$$F(s) = L\{f(x)\} = \int_0^\infty f(x) e^{-sx} dx,$$

where parameters s is a complex number $s = \sigma + iw$.
The inverse Laplace transform is given by the following complex integral:

$$f(x) = L^{-1}\{F(s)\} = \frac{1}{2\pi i} \lim_{T\to\infty} \int_{\gamma-iT}^{\gamma+iT} e^{sx}F(s)ds,$$

where the integration is done the vertical line $Re(s) = \gamma$ in the complex plane such that γ is greater than the real part of all singularities of $F(s)$ [3].
Definition 2.5 The Mittag-Leffler function plays a very important role in the solution of fractional-order differential equations [4,5].
More general Mittag-Leffler function with two parameters has the form

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \qquad z \in \mathbb{C}, \tag{2.1}$$

where $\alpha > 0, \beta > 0$, $\mathbb{C}$ denote the complex plane and $\Gamma(.)$ denote the Gamma function [8].
The Laplace transform of the Mittag-Leffler function in two parameters is [10]

$$L\{t^{\beta-1}E_{\alpha,\beta}(-\lambda t^{\alpha})\} = \frac{s^{\alpha-\beta}}{s^{\alpha}+\lambda}, \tag{2.2}$$

where $Res >| \lambda |^{\frac{1}{\alpha}}$.
Definition 2.6 The Laplace transform of the Capotu fractional derivative is defined as [7]

$$L\{D^{\alpha}f(t)\} = s^{\alpha}F(s) - \sum_{k=0}^{n-1} s^{(\alpha-k-1)}f^{(k)}(0), \qquad n-1 < \alpha \leq n. \tag{2.3}$$

3. Applications

In this section, two regular and singular fractional eigenvalue problems are solved using the Laplace transform.
Example 1 Consider the regular fractional eigenvalue problem

$$D^{1/2}y^{'}(x) + \lambda y(x) = 0, \qquad x \in (0,1), \tag{3.1}$$

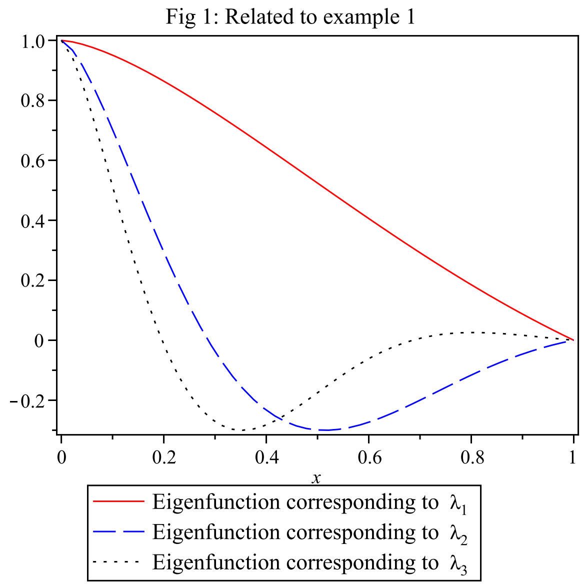
with the boundary conditions

$$y^{'}(0) = 0, \tag{3.2}$$

and

$$y(1) = 0. \tag{3.3}$$

Fig 1: Related to example 1



The Laplace transform (3.1) by using (2.3) yields

$$Y(s) = \frac{bs^{1/2}}{s^{3/2} + \lambda}, \tag{3.4}$$

where $y(0) = b$.
After taking inverse Laplace transform with respect to s in both sides of (3.4) and by using (2.2)

$$y(x) = bE_{3/2,1}(-\lambda x^{3/2}) = b \sum_{k=0}^{\infty} \frac{(-\lambda x^{3/2})^k}{\Gamma(\frac{3}{2}k + 1)}. \tag{3.5}$$

Now by (3.3) we have

$$b \sum_{k=0}^{\infty} \frac{(-\lambda)^k}{\Gamma(\frac{3}{2}k + 1)} = 0, \qquad b \neq 0. \tag{3.6}$$

Finally by solving (3.6) by using maple we can find the eigenvalues.
Therefore, the first three eigenvalues are identified as follows

$$\lambda_1 = 2.11027708, \qquad \lambda_2 = 13.76538087, \qquad \lambda_3 = 24.24337159.$$

The eigenfunctions corresponding to the above eigenvalues are shown in Fig 1.
Example 2 Consider the singular fractional eigenvalue problem

$$D^{1/2}y^{'}(x)+(\frac{1}{x}+\lambda)y(x)=0, \qquad x\in(0,1), \tag{3.7}$$

whit the boundary conditions

$$y(0)=0, \tag{3.8}$$

and

$$y^{'}(1)=0. \tag{3.9}$$

We have

$$xD^{1/2}y^{'}(x)+(1+\lambda x)y(x)=0. \tag{3.10}$$

By applying the Laplace transform to (3.10) gives

$$(s^{3/2}+\lambda)Y^{'}(s)+(\frac{3}{2}s^{1/2}-1)Y(s)=\frac{-1}{2}bs^{-3/2}, \tag{3.11}$$

where $y^{'}(0)=b$. Now we assume

$$Y(s)=\sum_{k=0}^{\infty}a_k s^{\frac{-k}{2}}. \tag{3.12}$$

The solution of this first-order linear differential equation (3.11) by series (3.12) we have

$$a_0=a_1=a_2=a_3=0, \quad a_4=1, \quad a_5=-1, \quad a_6=\frac{2}{3},$$

$$a_k=-\lambda a_{k-3}-\frac{2}{k-3}a_{k-1}, \qquad k\geq 7. \tag{3.13}$$

By applying the inverse Laplace transform to (3.12) gives

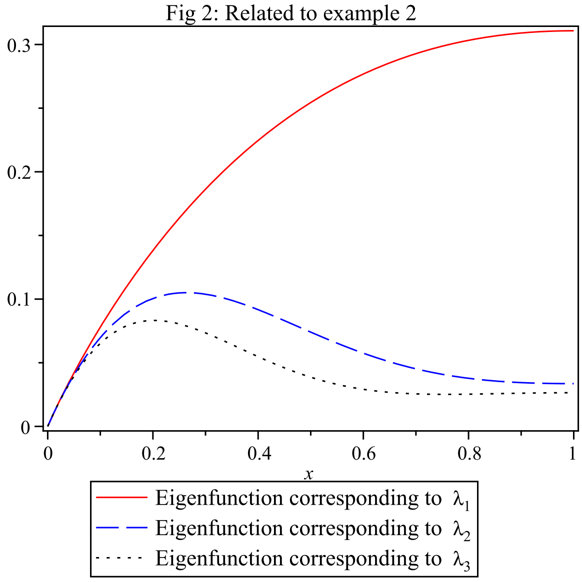
$$y(x)=\sum_{k=0}^{\infty}\frac{a_k}{\Gamma(\frac{k}{2})}x^{\frac{k}{2}-1}. \tag{3.14}$$

Now with the boundary conditions $y^{'}(1)=0$ and by using maple we can find the eigenvalues.
Therefore, the first three eigenvalues are identified as follows

$$\lambda_1=1.66091840, \qquad \lambda_2=13.55041954, \qquad \lambda_3=20.51439817.$$

The eigenfunctions corresponding to the above eigenvalues are shown in Figu 2.

Fig 2: Related to example 2



4. Conclusion

In this paper, We have proposed the analytical solution of the fractional Sturm-Liouville problems. Inverse Laplace transform method has been applied to numerically approximate the eigenvalues of the fractional Sturm-Liouville problems. The inverse Laplace transform method proved to be very efficient for computing the eigenvalues of the present problem.

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