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Chebyshev Spectral Collocation Method for Computing Numerical Solution of Telegraph Equation

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Abstract In this paper, the Chebyshev spectral collocation for one-dimensional linear hyperbolic telegraph equation is presented. Chebyshev spectral collocation method have become very useful in providing highly accurate solutions to partial differential equations. A straightforward implementation of these methods involves the use of spectral differentiation matrices. Firstly, we transform telegraph equation to system of partial differential equations with initial condition. Using Chebyshev differentiation matrices yields a system of ordinary differential equations. Secondly, we apply fourth order Runge-Kutta formula for the numerical integration of the system of ODEs. Numerical results verified the high accuracy of the new method, and its competitive ability compared with other newly appeared methods.

Keywords. Chebyshev spectral collocation method, telegraph equation, numerical results, Runge-Kutta formula.

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1. Introduction

Consider one dimensional linear hyperbolic telegraph equation:

$$\frac{\partial^2 u}{\partial t^2} + 2\alpha \frac{\partial u}{\partial t} + \beta^2 u = \frac{\partial^2 u}{\partial x^2} + f(x,t), \quad (x,t) \in [0,1] \times [0,1], \alpha > \beta \geq 0. \tag{1.1}$$

with initial conditions

$$u(x,0) = f_0(x), \;\; \frac{\partial u}{\partial t}(x,0) = f_1(x) \tag{1.2}$$

and boundary conditions

$$u(0,t) = g_0(t), u(1,t) = g_1(t), \quad t \geq 0. \tag{1.3}$$

Telegraph equation is commonly used in the study of wave propagation of electric signals in a cable transmission line and also in wave phenomena. Many researchers have used various numerical and analytical methods to solve the

Telegraph equation. Mohebbi and Dehaghan [19] studied high order compact solution to solve the telegraph equation. Gao and Chi [13] used unconditionally stable difference scheme for a one-space dimensional linear hyperbolic equation. Saadatmandi and Dehghan [20] developed a numerical solution based on Chebyshev Tau method. The authors of [22] used Legendre multiwavelet Galerkin method for solving the hyperbolic telegraph equation. Dehghan and Ghesmati [10] developed a numerical approach based on the truly meshless local weakstrong (MLWS) methods to deal with the second order two-space-dimensional telegraph equation. To solve the telegraph equation using the MLWS method, the conventional moving least squares (MLS) approximation is exploited in order to interpolate the solution of the equation. A time stepping scheme is employed to approximate the time derivative. Das and Gupta [9] used homotopy analysis method for solving fractional hyperbolic partial differential equations. By using initial values, the explicit solutions of telegraph equation for different particular cases have been derived. Abdou [1] used Adomian decomposition method for solving the telegraph equation in charged particle transport. The Authors of [15] developed a numerical technique for the solution of second order one dimensional linear hyperbolic equation. The method consists of expanding the required approximate solution as the elements of interpolating scaling functions. In their technique, by using the operational matrix of derivatives , they reduced the problem to a set of algebraic equations. Mohanty [17, 18] made investigations on the one-space-dimensional hyperbolic equations.

In [17], Mohanty carried over a new technique to solve the linear one-space-dimensional hyperbolic Eq. (1.1), which is unconditionally stable and is of second-order accuracy in both the time and space components. Also this author proposed in [18] a three level implicit unconditionally stable difference scheme [16] of second-order accuracy in both time and space variables for the solution of (1.1) with variable coefficients such that fictitious points are not needed at each time step along the boundary. Homotopy analysis method is developed in [14] to solve fractional IVPs. Borhanifar and Abazari [5] developed an unconditionally stable parallel difference scheme for telegraph equation. A numerical scheme is developed in [11] to solve the one-dimensional hyperbolic telegraph equation using collocation points [11] and approximating the solution using a thin plate splines radial basis function. Another numerical method is presented in [12] to solve the one-dimensional hyperbolic telegraph equation using Chebyshev cardinal functions. Also several test problems are given and the results of numerical experiments are compared with analytical solutions to confirm the good accuracy of the presented scheme. Differential transform method [4] is considered to solve telegraph equation. Using differential transform method, it is possible to find the exact solution or a closed

approximate solution of an equation. In this paper we use Chebyshev spectral collocation method (CSCM) to solve Eq. (1.1) with the initial and boundary conditions (1.2)-(1.3).
The outline of this paper is as follows. Section 1 contains a brief summary on telegraph equation. In section 2, we review some of the standard facts on Chebyshev spectral collocation method. In the third section, we develop the theory of transformation of telegraph equation to system of ordinary differential equations. In section 4, the numerical results of applying the method of this article on some test problems for the Eq. (1.1) are presented.

2. Chebyshev spectral collocation method

Consider a one-dimentional domain: $-1 \leq x \leq 1$. The domain of interest is discretized using the Gauss-Lobbato points defined as

$$\{\xi_j\} = \{\cos(\frac{j\pi}{N})\}_{j=0}^{N}. \tag{2.1}$$

We interpolate $u(x)$ by the polynomial $P(x)$ of degree at most N of the form:

$$P(x) = \sum_{j=0}^{N} \chi_j(x) u(\xi_j), \tag{2.2}$$

in the Chebyshev-Gauss-Lobbato (C-G-L) points with $\chi_j(x)$ for $j = 0(1)N$, are polynomial of degree at most N such that

$$\chi_j(\xi_k) = \delta_{jk}, \quad j,k = 0(1)N. \tag{2.3}$$

It can be shown that (see [2, 3, 6, 7, 8, 21]):

$$\chi_j(x) = \frac{(-1)^{j+1}(1-x^2)T'_N(x)}{\overline{\gamma}_j N^2 (x-\xi_j)}, \quad j = 0(1)N, \tag{2.4}$$

where

$$\overline{\gamma}_0 = \overline{\gamma}_N = 2, \quad \overline{\gamma}_j = 1, \quad j = 1(1)N-1,$$

and $T_N(x)$ the Chebyshev polynomial, i.e.,

$$T_N(x) = \cos(N \arccos x). \tag{2.5}$$

The values of derivative $\frac{d^kP}{dx^k}$, with $k = 1, 2, \cdots, p$ at the C-G-L points can be computed by

$$\widehat{\frac{d^kP}{dx^k}} = M^{(k)}\widehat{P} = M^k\widehat{P}, \tag{2.6}$$

where $\widehat{\cdot}$ labels vector, e.g., $\widehat{P} = (P(\xi_0), P(\xi_1), \cdots, P(\xi_N))^T$ and $M^{(1)}$ are the differentiation matrices. The entries of $M(M^{(1)})$ are

$$m_{kj} = -\frac{\gamma_k}{2\gamma_j}\frac{(-1)^{j+k}}{\sin((k+j)\frac{\pi}{2N})\sin((k-j)\frac{\pi}{2N})}, \quad k \neq j, \quad 0 \leq k, j \leq N, \tag{2.7}$$

$$m_{kj} = -\frac{1}{2}\cos(\frac{k\pi}{N})(1+\cot^2(\frac{k\pi}{N})), k = j, \quad k \neq 0, N, \tag{2.8}$$

$$m_{00} = -m_{NN} = \frac{2N^2+1}{6}. \tag{2.9}$$

As an alternative approach, the diagonal entries of M can be computed in the way that represents exactly the derivative of a constant [21]

$$m_{ii} = -\sum_{j=0, j\neq i}^{N} m_{ij}.$$

3. CSCM for telegraph equation

In this section, we outline the main step of our method to solve the telegraph equation (1.1) with initial conditions (1.2) and boundary conditions (1.3) by using CSCM. Set $\frac{\partial u}{\partial t}(x,t) = v(x,t)$. Then we can rewrite (1) as follows

$$\begin{array}{c} \frac{\partial v}{\partial t}(x,t) = -2\alpha v(x,t) - \beta^2 u(x,t) + \frac{\partial^2 u}{\partial x^2}(x,t) + f(x,t) \\ \frac{\partial u}{\partial t}(x,t) = v(x,t) \end{array} \tag{3.1}$$

with the initial conditions

$$u(x,0) = f_0(x), v(x,0) = f_1(x) \tag{3.2}$$

and the boundary conditions

$$u(0,t) = g_0(t), u(1,t) = g_1(t). \tag{3.3}$$

Now we describe the Chebyshev pseudospectral method for system of PDEs (3.1)-(3.3) to convert it to system of ODEs. For this let N be a nonnegative integer and denote by $\delta_j = \frac{1}{2}(1+\xi_j), \quad j = 0(1)N$, the Chebyshev-Gauss-Lobatto points in the interval $[0,1]$. We discretize (3.1) in space by the method of lines replacing $\frac{\partial u}{\partial x}(\delta_i, t), \frac{\partial^2 u}{\partial x^2}(\delta_i, t)$ by pseudospectral approximations given by

$$\frac{\partial u^k}{\partial x^k}(\delta_i, t) \approx 2\sum_{j=0}^{N} m_{ij}^k u(\delta_j, t), \quad i = 1(1)N-1, k = 1, 2 \tag{3.4}$$

and Here $M^{(k)}$ is differentiation matrix of order k. Substituting (3.4) into (3.3) and taking into account that

$$\begin{aligned} &u(\delta_N, t) = g_0(t), \\ &u(\delta_0, t) = g_1(t)), \end{aligned} \tag{3.5}$$

we obtain the following system of ODEs:

$$\begin{array}{c} \frac{\partial v}{\partial t}(\delta_i, t) = -2\alpha v(\delta_i, t) - \beta^2 u(\delta_i, t) \\ +2\sum_{j=1}^{N-1} m_{ij}^2 u(\delta_j, t) + 2(m_{i0}^2 g_1(t) + m_{iN}^2 g_0(t)) + f(\delta_i, t), \\ \frac{\partial u}{\partial t}(\delta_i, t) = v(\delta_i, t), \quad i = 1(1)N-1 \end{array} \tag{3.6}$$

with the initial conditions

$$u(\delta_i, 0) = f_0(\delta_i), v(\delta_i, 0) = f_1(\delta_i), \quad i = 1(1)N-1. \tag{3.7}$$

We can write the equations (3.6)-(3.7) in the matrix form as follows

$$\frac{d\overline{U}}{dt} = A\overline{U} + G, \tag{3.8}$$

where

$$\overline{U} = \left[\begin{array}{c} U \\ V \end{array}\right], A = \left[\begin{array}{cc} o_{N-1,N-1} & I_{N-1,N-1} \\ B & C \end{array}\right], G = \left[\begin{array}{c} o_{N-1,1} \\ D \end{array}\right],$$

$$U = \left[\begin{array}{c} u(\delta_1, t) \\ u(\delta_2, t) \\ \vdots \\ u(\delta_{N-1}, t) \end{array}\right], V = \left[\begin{array}{c} v(\delta_1, t) \\ v(\delta_2, t) \\ \vdots \\ v(\delta_{N-1}, t) \end{array}\right],$$

and

$$B = \left[\begin{array}{cccc} 2(m_{11}^2 - \beta^2) & 2m_{12}^2 & \dots & 2m_{1,N-1}^2 \\ 2m_{21}^2 & 2(m_{22}^2 - \beta^2) & \dots & 2m_{2,N-1}^2 \\ \vdots & \vdots & & \vdots \\ 2m_{N-1,1}^2 & 2m_{N-1,2}^2 & \dots & 2(m_{N-1,N-1}^2 - \beta^2) \end{array}\right]_{N-1,N-1},$$

and

$$C = \left[\begin{array}{cccc} -2\alpha & 0 & \dots & 0 \\ 0 & -2\alpha & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & -2\alpha \end{array}\right]_{N-1,N-1},$$

$$D = \begin{bmatrix} 2(m_{10}^{(2)} g_1(t) + m_{1,N}^{(2)} g_0(t)) + f(\delta_1, t) \\ 2(m_{20}^{(2)} g_1(t) + m_{2,N}^{(2)} g_0(t)) + f(\delta_2, t) \\ \vdots \\ 2(m_{N-1,0}^{(2)} g_1(t) + m_{N-1,N}^{(2)} g_0(t)) + f(\delta_{N-1}, t) \end{bmatrix},$$

with the initial condition

$$\overline{U}(0) = \begin{bmatrix} U(0) \\ V(0) \end{bmatrix}. \tag{3.9}$$

We apply fourth order Runge-Kutta formula for the numerical integration of the system of ODEs (3.8) with initial conditions (3.9).

4. Numerical results

Example 1. We consider the Eq. (1.1) with the following conditions [1]:

$$\begin{aligned} &f_0(x) = \sinh(x), \\ &f_1(x) = -2\sinh(x), \\ &g_0(t) = 0, \\ &g_1(t) = e^{-2t}\sinh(1), \\ &f(x,t) = (3 - 4\alpha + \beta^2)e^{-2t}\sinh(x). \end{aligned} \tag{4.1}$$

The exact solution is given by

$$u(x,t) = e^{-2t}\sinh(x). \tag{4.2}$$

In this section, we give some computational results numerical experiments with the method based on the preceding sections. To show the efficiency of the present method for our problems in comparison with the exact solution we calculate the maximum error $||u||_\infty$ defined by

$$||u||_\infty = \max\{|u_{numer} - u_{exact}| : 0 \le i \le N\}.$$

Where u_{numer} and u_{exact} are the numerical and exact solution, respectively. Numerical computations were carried out in Matlab.
Table 1 show the absolute error using the technique presented in the previous section with $\Delta t = 0.001, N = 16, \alpha = 20$ and $\beta = 10$ for $t = 0.5, 1, 1.5, 2$.

Table 1. Absolute error for $u(\delta_j, t)$ with $\Delta t = 0.001, N = 16, \alpha = 20$ and $\beta = 10$ for $t = 0.5, 1, 1.5, 2$.

δ/t	0.5	1	1.5	2
δ_2	0.0020	$9.7848e-4$	$4.0835e-4$	$1.6057e-4$
δ_6	0.0024	0.0015	$7.3220e-4$	$3.1182e-4$
δ_{10}	$9.8950e-4$	$6.4268e-4$	$3.0918e-4$	$1.3295e-4$
δ_{14}	$1.2012e-4$	$7.8022e-5$	$3.7535e-5$	$1.6140e-5$

In Fig. 1, we plot exact solution and numerical solution at $t = 1, N = 128, \beta = 10, \alpha = 20$ and $\Delta t = 0.001$ for Example 1.

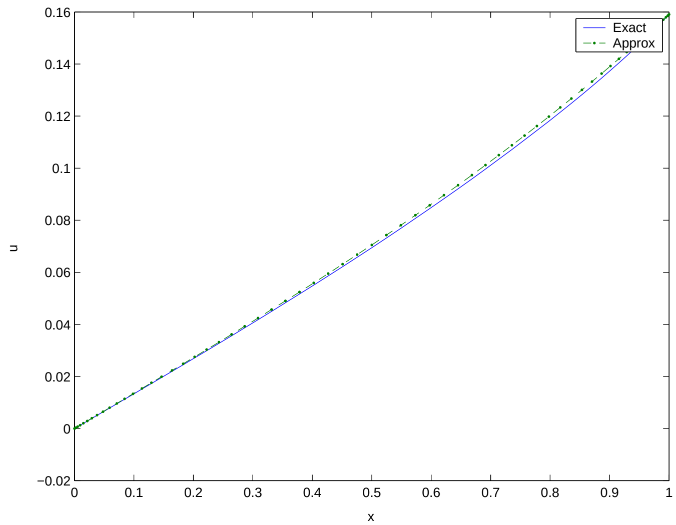

FIGURE 1. Exact solution and numerical solution at $t = 1, N = 128, \beta = 10, \alpha = 20$ and $\Delta t = 0.001$ for Example 1.

In Fig. 2, we plot max error at $t = 0.5(0.5)2, N = 5(5)50, \beta = 1, \alpha = 5$ and $\Delta t = 0.001$ for Example 1.

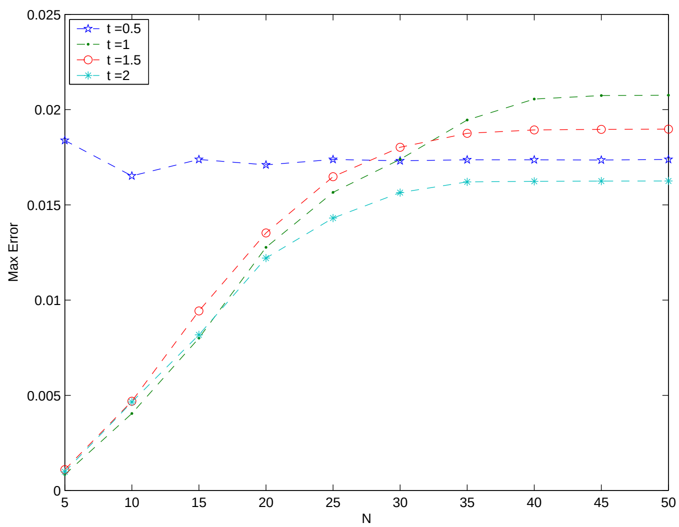

FIGURE 2. Max error at $t = 0.5(0.5)2, N = 5(5)50, \beta = 1, \alpha = 5$ and $\Delta t = 0.001$ for Example 1.

In Fig. 3, we plot max error at $t = 1, N = 5(5)50, \beta = 1$ and $\Delta t = 0.001$ for Example 1.

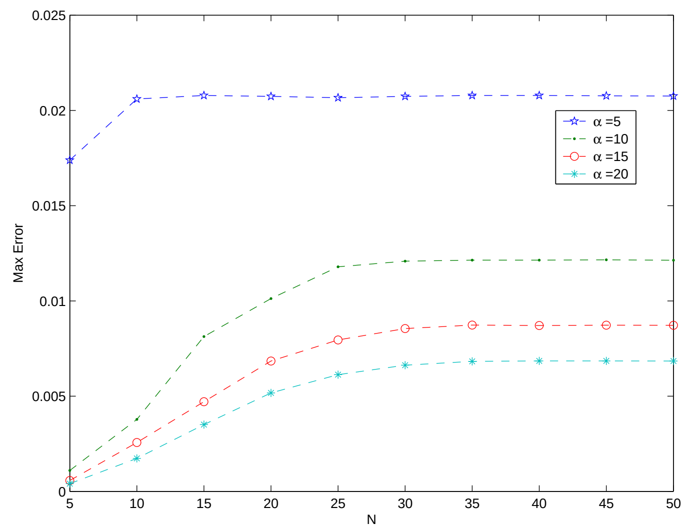

FIGURE 3. Max error at $t = 1, N = 5(5)50, \beta = 1$ and $\Delta t = 0.001$ for Example 1.

In Fig. 4, we plot log(max error) at $t = 1, N = 5(5)50, \alpha = 2$ and $\Delta t = 0.001$ for Example 1.

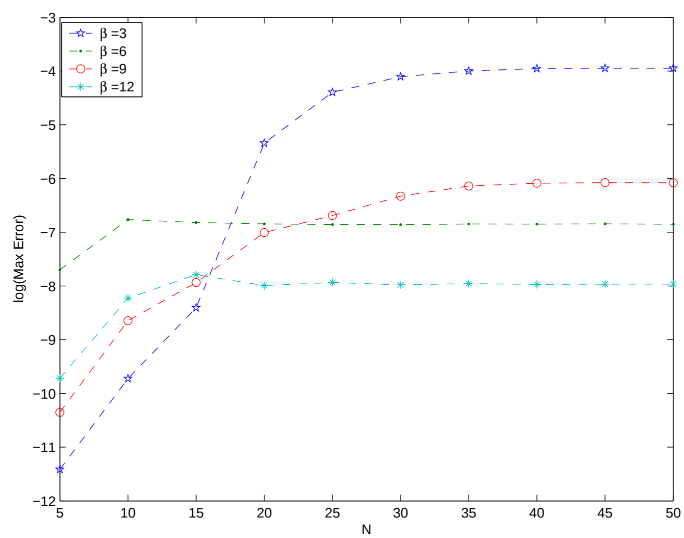

FIGURE 4. log(Max error) at $t = 1, N = 5(5)50, \alpha = 2$ and $\Delta t = 0.001$ for Example 1.

In Fig. 5, we plot log(absolute error) at $t = 1, N = 16, \alpha = 2$ and $\Delta t = 0.001$ for Example 1.

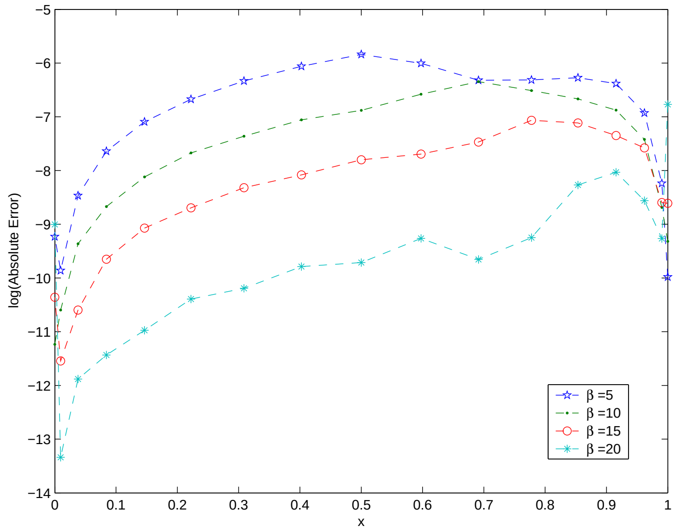

FIGURE 5. log(absolute error) at $t = 1, N = 16, \alpha = 2$ and $\Delta t = 0.001$ for Example 1.

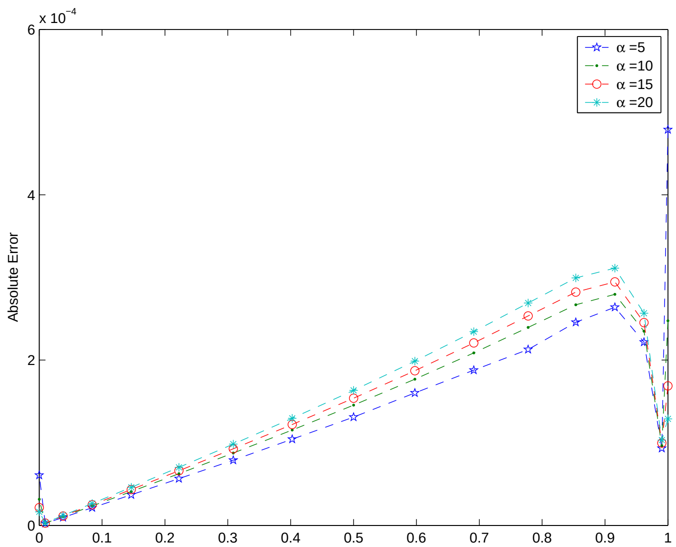

FIGURE 6. Absolute error at $t = 1, N = 16, \beta = 20$ and $\Delta t = 0.001$ for Example 1.

In Fig. 6, we plot absolute error at $t = 1, N = 16, \beta = 20$ and $\Delta t = 0.001$ for Example 1.

Example 2. We consider the Eq. (1.1) with the following conditions:

$$\begin{aligned} &f_0(x) = \sin(x), \\ &f_1(x) = 0, \\ &g_0(t) = 0, \\ &g_1(t) = \cos(t)\sinh(1), \\ &f(x,t) = -2\alpha\sin(t)\sin(x) + \beta^2)\cos(t)\sin(x). \end{aligned} \tag{4.3}$$

The exact solution is given by

$$u(x,t) = \cos(t)\sin(x). \tag{4.4}$$

Table 2 show the absolute error using the technique presented in the previous section with $\Delta t = 0.001, N = 16, \alpha = 20$ and $\beta = 10$ at $t = 0.5, 1, 1.5, 2$ for Example 2.

Table 2. Absolute error for $u(\delta_j, t)$ with $\Delta t = 0.0001, N = 64, \alpha = 20$ and $\beta = 10$ at $t = 0.5, 1, 1.5, 2$ for Example 2.

δ/t	0.5	1	1.5	2
δ_2	$2.3747e-4$	$1.8608e-4$	$7.0089e-5$	$6.6330e-5$
δ_6	0.0017	0.0014	$5.8266e-4$	$4.1998e-4$
δ_{10}	0.0032	0.0028	0.0013	$6.0133e-4$
δ_{14}	0.0038	0.0036	0.0019	$4.9635e-4$

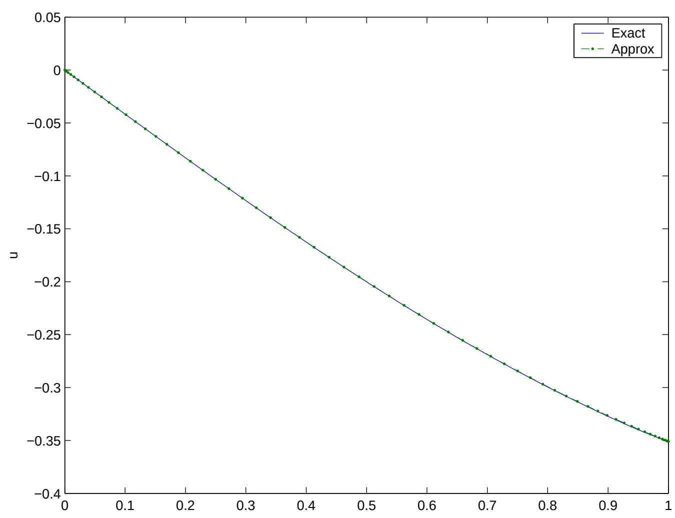

FIGURE 7. Exact solution and numerical solution at $t = 2, N = 64, \beta = 10, \alpha = 20$ and $\Delta t = 0.0001$ for Example 2.

In Fig. 7, we plot exact solution and numerical solution at $t = 2, N = 64, \beta = 10, \alpha = 20$ and $\Delta t = 0.0001$ for Example 2. In Fig. 8, we plot max error at $t = 0.5(0.5)2, N = 5(5)50, \beta = 1, \alpha = 5$ and $\Delta t = 0.001$ for Example 2.

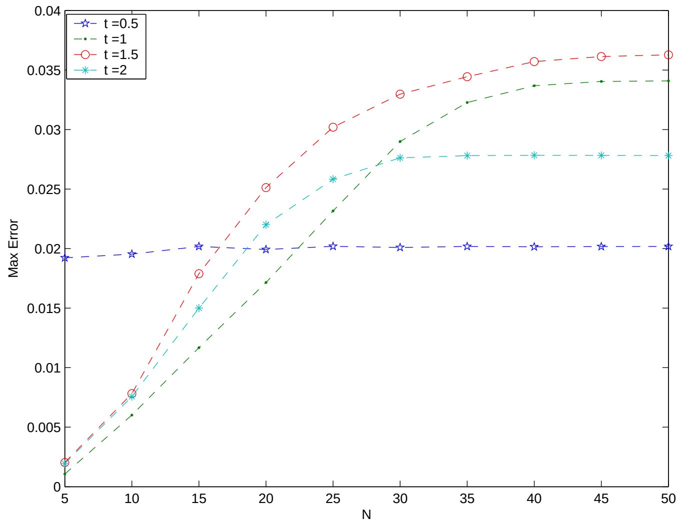

FIGURE 8. Max error at $t = 0.5(0.5)2, N = 5(5)50, \beta = 1, \alpha = 5$ and $\Delta t = 0.001$ for Example 2.

In Fig. 9, we plot max error at $t = 1, N = 5(5)50, \beta = 1$ and $\Delta t = 0.001$ for Example 2.

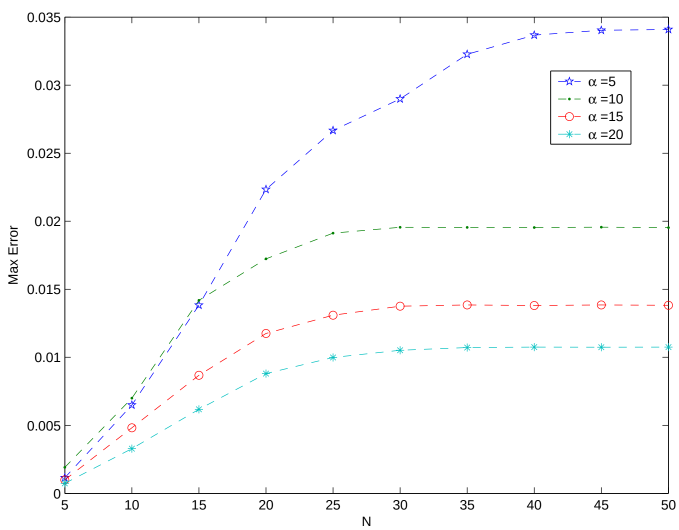

FIGURE 9. Max error at $t = 1, N = 5(5)50, \beta = 1$ and $\Delta t = 0.001$ for Example 2.

In Fig. 10, we plot log(max error) at $t = 1, N = 5(5)50, \alpha = 2$ and $\Delta t = 0.001$ for Example 2.

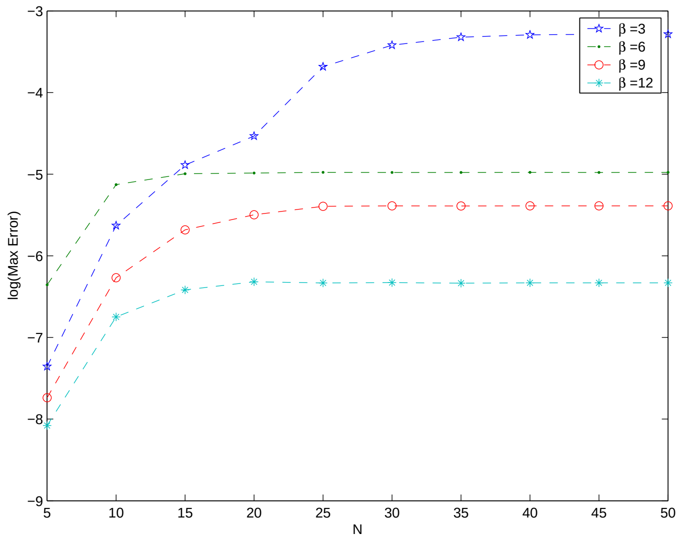

FIGURE 10. log(Max error) at $t = 1, N = 5(5)50, \alpha = 2$ and $\Delta t = 0.001$ for Example 2.

In Fig. 11, we plot log(absolute error) at $t = 1, N = 16, \alpha = 2$ and $\Delta t = 0.001$ for Example 2.

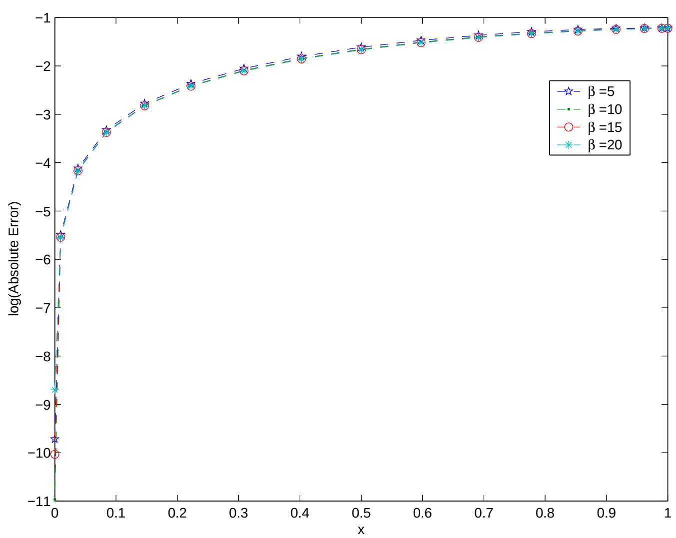

FIGURE 11. log(absolute error) at $t = 1, N = 16, \alpha = 2$ and $\Delta t = 0.001$ for Example 2.

5. Conclusions

In this paper, a Chebyshev spectral collocation semi-discretization in space is applied to numerical solution of telegraph equation. We describe behavior of telegraph equation for various values of α and β at long time. Also we describe behavior of telegraph equation for various values of N. The obtained results show that this approach can solve the problem effectively.

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