



The extended homogeneous balance method and exact 1-soliton solutions of the Maccari system

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Abstract The extended homogeneous balance method is used to construct exact traveling wave solutions of the Maccari system, in which the homogeneous balance method is applied to solve the Riccati equation and the reduced nonlinear ordinary differential equation. Many exact traveling wave solutions of the Maccari system equation are successfully obtained.

Keywords. Extended homogeneous balance method; Maccari system; Riccati equation; Soliton-like solution; Periodic-like solution.

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1. INTRODUCTION

The study of nonlinear partial differential equations (PDEs) appear everywhere in applied mathematics and theoretical physics including engineering sciences and biological sciences. These equations play a key role in describing key scientific phenomena [1-15]. Many powerful methods for obtaining the exact solutions of nonlinear PDEs have been presented, such as homogeneous balance method [1-4], tanh method [5-7], Jacobian elliptic function expansion method [8], first integral method [9-11] and so on.

One of the most effective direct methods to develop the traveling wave solution of nonlinear PDEs is the homogeneous balance method [1]. This method has been successfully applied to obtain exact solutions for a variety of nonlinear PDEs [2-4]. In this paper, we use the homogeneous balance method to solve the Riccati equation $\phi' = a\phi^2 + b\phi + c$ and the reduced nonlinear ordinary differential equation. It makes use of the homogeneous balance method more extensively. Our method is called the extended homogeneous balance method since we have used the homogeneous balance method twice. Our method can be used to find more exact traveling wave solutions of a given nonlinear partial differential equation; the Maccari system is chosen to illustrate this approach.

2. DISCRIPTION OF METHOD

We will apply the extended homogeneous balance method to construct the exact 1-soliton solutions of Maccari system [13, 15]

$$\begin{aligned} iu_t + u_{xx} + uv &= 0, \\ v_t + v_y + (|u|^2)_x &= 0. \end{aligned} \quad (2.1)$$

In order to find travelling wave solutions of Eq. (1.1), we set

$$u(x, y, t) = e^{i\theta} u(\xi), \quad v(x, y, t) = v(\xi), \quad \theta = px + qy + rt, \quad \xi = x + \alpha y - 2pt, \quad (2.2)$$

where p, q, r and α are constants.

Substituting (1.2) into Eq. (1.1), then Eq. (1.1) is reduced to the following nonlinear ordinary differential equation:

$$\begin{aligned} -(r + p^2)u + u'' + uv &= 0, \\ (\alpha - 2p)v' + 2uu' &= 0. \end{aligned} \quad (2.3)$$

We now seek the solutions of Eq. (1.3) in the form

$$u = \sum_{l=0}^n a_l \phi^l, \quad (2.4)$$

$$v = \sum_{l=0}^m b_l \phi^l, \quad (2.5)$$

where a_l, b_l are constants to be determined later and ϕ satisfy the following Riccati equation

$$\phi' = a\phi^2 + b\phi + c, \quad (2.6)$$

where a, b and c are constants.

Balancing the highest order derivative terms with nonlinear terms u'' with uv and v' with uu' in Eq. (1.3) gives leading order $n = 1, m = 2$.

Therefore we use the ansatz

$$u = a_0 + a_1 \phi, \quad (2.7)$$

$$v = b_0 + b_1 \phi + b_2 \phi^2, \quad (2.8)$$

where a_0, a_1, b_0, b_1 and b_2 are constants to be determined and ϕ satisfy Eq. (1.6). Substituting Eqs. (1.7)-(1.8) into Eq. (1.3), and equating the coefficients of like powers of ϕ^l ($l = 0, 1, 2, 3$) to zero yields the system of algebraic equations in a_0, a_1, b_0, b_1 and b_2

$$\begin{aligned} 2a^2 a_1 + a_1 b_2 &= 0, \\ 3aba_1 + a_0 b_2 + a_1 b_1 &= 0, \\ a_1 b_0 + 2aca_1 + b^2 a_1 - (r + p^2)a_1 + a_0 b_1 &= 0, \\ -(r + p^2)a_0 + bca_1 + a_0 b_0 &= 0, \\ 2(\alpha - 2p)ab_2 + 2aa_1^2 &= 0, \\ (\alpha - 2p)ab_1 + 2(\alpha - 2p)bb_2 + 2aa_0 a_1 + 2ba_1^2 &= 0, \end{aligned}$$



$$\begin{aligned}
 (\alpha - 2p)bb_1 + 2(\alpha - 2p)cb_2 + 2ba_0a_1 + 2ca_1^2 &= 0, \\
 (\alpha - 2p)cb_1 + 2ca_0a_1 &= 0,
 \end{aligned}
 \tag{2.9}$$

for which, with the aid of Maple, we get the following solution:

$$\begin{aligned}
 a_0 &= \pm \frac{b}{2} \sqrt{(2\alpha - 4p)}, a_1 = \pm a \sqrt{(2\alpha - 4p)}, \\
 b_0 &= r + p^2 - 2ac, b_1 = -2ab, b_2 = -2a^2.
 \end{aligned}
 \tag{2.10}$$

It is to be noted that the Riccati equation (1.6) can be solved using the homogeneous balance method as follows:

Case I. Let $\phi = \sum_{l=0}^m b_l \tanh^l \xi$. Balancing ϕ' with ϕ^2 leads to

$$\phi = b_0 + b_1 \tanh \xi.
 \tag{2.11}$$

Substituting Eq. (1.11) into Eq. (1.6), we obtain the following solution of Eq. (1.6):

$$\phi = -\frac{1}{2a}(b + 2 \tanh \xi), ac = \frac{b^2}{4} - 1.
 \tag{2.12}$$

Substituting Eqs. (1.10) and (1.12) into (1.2), (1.7) and (1.8), we have the following travelling wave solutions of the Maccari system (1.1):

$$\begin{aligned}
 u(x, y, t) &= \pm \sqrt{(2\alpha - 4p)} e^{i(px+qy+rt)} \tanh(x + \alpha y - 2pt), \\
 v(x, y, t) &= r + p^2 + \frac{b^2}{2} - 2ac - 2 \tanh^2(x + \alpha y - 2pt),
 \end{aligned}
 \tag{2.13}$$

where $ac = \frac{b^2}{4} - 1$.

Similarly, let $\phi = \sum_{l=0}^m b_l \coth^l \xi$, then we obtain the following travelling wave solutions of the Maccari system (1.1):

$$\begin{aligned}
 u(x, y, t) &= \pm \sqrt{(2\alpha - 4p)} e^{i(px+qy+rt)} \coth(x + \alpha y - 2pt), \\
 v(x, y, t) &= r + p^2 + \frac{b^2}{2} - 2ac - 2 \coth^2(x + \alpha y - 2pt),
 \end{aligned}
 \tag{2.14}$$

where $ac = \frac{b^2}{4} - 1$.

Case II. From [12], when $a = 1, b = 0$, the Riccati Eq. (1.6) has the following solutions:

$$\begin{aligned}
 \phi &= -\sqrt{-c} \tanh(\sqrt{-c}\xi), c < 0, \\
 \phi &= -\frac{1}{\xi}, c = 0, \\
 \phi &= \sqrt{c} \tan(\sqrt{c}\xi), c > 0.
 \end{aligned}
 \tag{2.15}$$

From (1.7), (1.8), (1.10) and (1.15), we have the following travelling wave solutions of the Maccari system (1.1):



When $c < 0$, we have

$$\begin{aligned} u(x, y, t) &= \mp \sqrt{-c(2\alpha - 4p)} e^{i(px+qy+rt)} \tanh(\sqrt{-c}(x + \alpha y - 2pt)), \\ v(x, y, t) &= r + p^2 - 2c + 2c \tanh^2(\sqrt{-c}(x + \alpha y - 2pt)). \end{aligned} \quad (2.16)$$

When $c = 0$, we have

$$\begin{aligned} u(x, y, t) &= \mp e^{i(px+qy+rt)} \frac{\sqrt{(2\alpha - 4p)}}{x + \alpha y - 2pt}, \\ v(x, y, t) &= r + p^2 - \frac{2}{(x + \alpha y - 2pt)^2}. \end{aligned} \quad (2.17)$$

When $c > 0$, we have

$$\begin{aligned} u(x, y, t) &= \pm \sqrt{c(2\alpha - 4p)} e^{i(px+qy+rt)} \tan(\sqrt{c}(x + \alpha y - 2pt)), \\ v(x, y, t) &= r + p^2 - 2c - 2c \tan^2(\sqrt{c}(x + \alpha y - 2pt)). \end{aligned} \quad (2.18)$$

Case III. We suppose that the Riccati Eq. (1.6) has solution of the form

$$\phi = A_0 + \sum_{l=1}^m \sinh^{l-1}(A_l \sinh \omega + B_l \cosh \omega),$$

where $\frac{d\omega}{d\xi} = \sinh \omega$ or $\frac{d\omega}{d\xi} = \cosh \omega$. It is easy to find that $m = 1$ by balancing ϕ' and ϕ^2 . So we choose

$$\phi = A_0 + A_1 \sinh \omega + B_1 \cosh \omega, \quad (2.19)$$

when $\frac{d\omega}{d\xi} = \sinh \omega$, we substitute (1.19) and $\frac{d\omega}{d\xi} = \sinh \omega$, into Eq. (1.6) and set the coefficients of $\sinh^k \omega \cosh^l \omega$ ($k = 0, 1, 2$; $l = 0, 1$) to zero. A set of algebraic equations is obtained as follows:

$$\begin{aligned} aA_0^2 + aB_1^2 + bA_0 + c &= 0, \\ 2aA_0A_1 + bA_1 &= 0, \\ aA_1^2 + aB_1^2 - B_1 &= 0, \\ 2aA_0B_1 + bB_1 &= 0, \end{aligned}$$

$$2aA_1B_1 + A_1 = 0, \quad (2.20)$$

for which, we have the following solutions:

$$A_0 = -\frac{b}{2a}, \quad A_1 = 0, \quad B_1 = \frac{1}{a}, \quad (2.21)$$

where $c = \frac{b^2 - 4}{4a}$, and

$$A_0 = -\frac{b}{2a}, \quad A_1 = \pm \sqrt{\frac{1}{2a}}, \quad B_1 = \frac{1}{2a}, \quad (2.22)$$



where $c = \frac{b^2-1}{4a}$.

To $\frac{d\omega}{d\xi} = \sinh \omega$, we have

$$\sinh \omega = -\operatorname{csch} \xi, \quad \cosh \omega = -\operatorname{coth} \xi. \tag{2.23}$$

From (1.21)-(1.23), we obtain

$$\phi = -\frac{b + 2 \operatorname{coth} \xi}{2a}, \tag{2.24}$$

where $c = \frac{b^2-4}{4a}$, and

$$\phi = -\frac{b \pm \operatorname{csch} \xi + \operatorname{coth} \xi}{2a}, \tag{2.25}$$

where $c = \frac{b^2-1}{4a}$.

From Eqs. (1.7), (1.8), (1.10), (1.24) and (1.25), we get the travelling wave solutions of the Maccari system (1.1) in the following form:

$$\begin{aligned} u(x, y, t) &= \pm \sqrt{(2\alpha - 4p)} e^{i(px+qy+rt)} \operatorname{coth}(x + \alpha y - 2pt), \\ v(x, y, t) &= r + p^2 + \frac{b^2}{2} - 2ac - 2 \operatorname{coth}^2(x + \alpha y - 2pt), \end{aligned} \tag{2.26}$$

where $c = \frac{b^2-4}{4a}$, and

$$\begin{aligned} u(x, y, t) &= \pm \frac{\sqrt{(2\alpha - 4p)}}{2} e^{i(px+qy+rt)} \{ \operatorname{coth}(x + \alpha y - 2pt) \pm \operatorname{csch}(x + \alpha y - 2pt) \}, \\ v(x, y, t) &= r + p^2 + \frac{b^2}{2} - 2ac - \frac{1}{2} \{ \operatorname{coth}(x + \alpha y - 2pt) \pm \operatorname{csch}(x + \alpha y - 2pt) \}^2, \end{aligned} \tag{2.27}$$

where $c = \frac{b^2-1}{4a}$.

Similarly, when $\frac{d\omega}{d\xi} = \cosh \omega$, we obtain the following traveling wave solutions of the Maccari system (1.1) in the following form:

$$u(x, y, t) = \pm \sqrt{(2\alpha - 4p)} e^{i(px+qy+rt)} \cot(x + \alpha y - 2pt), \tag{2.28}$$

$$v(x, y, t) = r + p^2 + \frac{b^2}{2} - 2ac - 2 \cot^2(x + \alpha y - 2pt), \tag{2.29}$$

where $c = \frac{b^2-4}{4a}$, and

$$u(x, y, t) = \pm \frac{\sqrt{(2\alpha - 4p)}}{2} e^{i(px+qy+rt)} \{ \cot(x + \alpha y - 2pt) \pm \operatorname{csc}(x + \alpha y - 2pt) \}, \tag{2.30}$$

$$v(x, y, t) = r + p^2 + \frac{b^2}{2} - 2ac - \frac{1}{2} \{ \cot(x + \alpha y - 2pt) \pm \operatorname{csc}(x + \alpha y - 2pt) \}^2, \tag{2.31}$$



where $c = \frac{b^2-1}{4a}$.

Case IV. From [14], the Riccati Eq. (1.6) admits the following exact solution:

$$\phi(\xi) = -\frac{b}{2a} - \frac{\theta}{2a} \tanh\left(\frac{\theta}{2}(\xi + \xi_0)\right) \quad (2.32)$$

and

$$\phi(\xi) = -\frac{b}{2a} - \frac{\theta}{2a} \tanh\left(\frac{\theta}{2}\xi\right) + \frac{\operatorname{sech}\left(\frac{\theta}{2}\xi\right)}{C \cosh\left(\frac{\theta}{2}\xi\right) - \frac{2a}{\theta} \sinh\left(\frac{\theta}{2}\xi\right)} \quad (2.33)$$

where $\theta^2 = b^2 - 4ac > 0$ and C is a constant of integration.

From Eqs. (1.7), (1.8), (1.10), (1.32) and (1.33), we obtain the following new traveling wave solutions of the Maccari system (1.1):

$$\begin{aligned} u(x, y, t) &= \mp \frac{\theta \sqrt{(2\alpha - 4p)}}{2} e^{i(px+qy+rt)} \tanh\left(\frac{\theta}{2}(x + \alpha y - 2pt + \xi_0)\right), \\ v(x, y, t) &= r + p^2 - 2ac + 2b \left\{ \frac{b}{2} - \frac{\theta}{2} \tanh\left(\frac{\theta}{2}(x + \alpha y - 2pt + \xi_0)\right) \right\} \\ &\quad - 2 \left\{ \frac{b}{2} + \frac{\theta}{2} \tanh\left(\frac{\theta}{2}(x + \alpha y - 2pt + \xi_0)\right) \right\}^2 \end{aligned} \quad (2.34)$$

and

$$u(x, y, t) = \pm \sqrt{(2\alpha - 4p)} e^{i(px+qy+rt)} \left\{ \frac{b}{2} + a \left(-\frac{b}{2a} - \frac{\theta}{2a} \tanh\left(\frac{\theta}{2}\xi\right) + \frac{\operatorname{sech}\left(\frac{\theta}{2}\xi\right)}{C \cosh\left(\frac{\theta}{2}\xi\right) - \frac{2a}{\theta} \sinh\left(\frac{\theta}{2}\xi\right)} \right) \right\}, \quad (2.35)$$

$$\begin{aligned} v(x, y, t) &= r + p^2 - 2ac - 2ab \left\{ -\frac{b}{2a} - \frac{\theta}{2a} \tanh\left(\frac{\theta}{2}\xi\right) + \frac{\operatorname{sech}\left(\frac{\theta}{2}\xi\right)}{C \cosh\left(\frac{\theta}{2}\xi\right) - \frac{2a}{\theta} \sinh\left(\frac{\theta}{2}\xi\right)} \right\} \\ &\quad - 2a^2 \left\{ -\frac{b}{2a} - \frac{\theta}{2a} \tanh\left(\frac{\theta}{2}\xi\right) + \frac{\operatorname{sech}\left(\frac{\theta}{2}\xi\right)}{C \cosh\left(\frac{\theta}{2}\xi\right) - \frac{2a}{\theta} \sinh\left(\frac{\theta}{2}\xi\right)} \right\}^2 \end{aligned} \quad (2.36)$$

where $\theta^2 = b^2 - 4ac > 0$, $\xi = x + \alpha y - 2pt$ and C is a constant of integration.

Case V. From [14], when $c = 0$, then the Riccati Eq. (1.6) reduces to the Bernoulli equation

$$\phi' = a\phi^2 + b\phi. \quad (2.37)$$

The solution of the Bernoulli Eq. (1.36) can be written in the following form [14]:

$$\phi(\xi) = b \left\{ \frac{\cosh(b(\xi + \xi_0)) + \sinh(b(\xi + \xi_0))}{1 - a \cosh(b(\xi + \xi_0)) - a \sinh(b(\xi + \xi_0))} \right\}, \quad (2.38)$$

From Eqs. (1.7), (1.8), (1.10) and (1.37), we obtain the following new traveling wave solutions of the Maccari system (1.1):



$$\begin{aligned}
u(x, y, t) &= \pm \sqrt{(2\alpha - 4p)} e^{i(px+qy+rt)} \left\{ \frac{b}{2} + ab \left\{ \frac{\cosh(b(\xi + \xi_0)) + \sinh(b(\xi + \xi_0))}{1 - a \cosh(b(\xi + \xi_0)) - a \sinh(b(\xi + \xi_0))} \right\} \right\}, \\
v(x, y, t) &= r + p^2 - 2ac - 2ab^2 \left\{ \frac{\cosh(b(\xi + \xi_0)) + \sinh(b(\xi + \xi_0))}{1 - a \cosh(b(\xi + \xi_0)) - a \sinh(b(\xi + \xi_0))} \right\} \\
&\quad - 2a^2 b^2 \left\{ \frac{\cosh(b(\xi + \xi_0)) + \sinh(b(\xi + \xi_0))}{1 - a \cosh(b(\xi + \xi_0)) - a \sinh(b(\xi + \xi_0))} \right\}^2, \tag{2.39}
\end{aligned}$$

where $\xi = x + \alpha y - 2pt$.

In summary we have used the homogeneous balance method to obtain many travelling wave solutions of the Maccari system.

We now summarize the key steps as follows:

Step I: For a given nonlinear evolution equation

$$P(u, u_t, u_x, u_{xt}, u_{tt}, \dots) = 0, \tag{2.40}$$

we consider its travelling wave solutions $u(x, t) = e^{i(\alpha x + \beta t)} u(\xi)$, $\xi = kx + lt$, then Eq. (1.39) is reduced to a nonlinear ordinary differential equation

$$Q(u, u', u'', u''', \dots) = 0, \tag{2.41}$$

where a prime denotes $\frac{d}{d\xi}$.

Step II: For a given ansatz equation (for example, the ansatz equation is $\phi' = a\phi^2 + b\phi + c$ in this paper), the form of u is decided and the homogeneous balance method is used on Eq. (1.40) to find the coefficients of u .

Step III: The homogeneous balance method is used to solve the ansatz equation.

Step IV: Finally, the travelling wave solutions of Eq. (1.39) are obtained by combining steps II and III.

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