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Numerical solution of Burgers' equation with nonlocal boundary condition: Use of Keller-Box scheme

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Abstract

In this paper, we transform the given nonlocal boundary condition problem into a manageable local equation. By introducing an additional transformation of the variables, we can simplify this equation into conformable Burgers' equation. Thus, the Keller Box method is used as a numerical scheme to solve the equation. A comparison is made between numerical results and the analytic solution to validate the results of our proposed method.

Keywords. Nonlocal boundary condition, Burgers' Equation, Keller-Box scheme.

2010 Mathematics Subject Classification. 65L05, 34K06, 34K28.

1. Introduction

When solving partial differential equations (PDEs), a non-local boundary condition is a special type of boundary condition that relies on solution values at points beyond the boundary. Unlike local boundary conditions, which only consider solution values at the boundary, non-local boundary conditions require data from a wider region of the domain to determine the behavior of the solution at the boundary [22, 34, 37]. This can be particularly useful when modelling physical phenomena with long-range interactions or memory effects, such as heat transfer in materials, chemical diffusion, thermoelasticity, inverse problems, and biological processes [10, 24]. In physics, for example, non-local boundary conditions can be used to model phenomena such as heat conduction in materials with memory, where the current temperature at a point depends not only on the temperatures of neighboring points, but also on the temperatures at that point at previous times. J. Cannon [9] pioneered the consideration of integral boundary conditions as an alternative to the classical Dirichlet or Neumann boundary conditions for one-dimensional parabolic equations. In addition, further research into non-classical problems with integral boundary conditions in the context of various evolution equations has been explored in [4, 7, 8, 11, 23, 25, 30–32]. Siddique [33] aims to improve the efficiency of the Crank-Nicolson numerical scheme when solving two-dimensional parabolic partial differential equations with nonlocal boundary conditions. A numerical method based on Padé approximations of the matrix exponential is utilised. The numeric results clearly demonstrate the enhancement attained by employing the improved Crank-Nicolson numerical technique with nonlocal boundary conditions. Berikelashvili and Khomeiri [4] examine a nonlocal boundary-value problem for the Poisson equation in a rectangular domain. One pair of adjacent sides of the rectangle have Dirichlet and Neumann conditions imposed, while integral constraints replace traditional boundary conditions on the remaining pairs of sides. Predictions of the solution can be made using the energy inequality method. Furthermore, a computation of the discretization error estimate is conducted that corresponds with the expected smoothness characteristics of the solution. Islam et al. [17] investigated two numerical techniques for solving the two-dimensional Poisson equation under different nonlocal boundary conditions. Initially, they employed Haar wavelets and a collocation method, which included a novel approach to approximating mixed derivatives. The second technique utilizes a meshless method-based

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on different types of radial basis functions (RBFs). Furthermore, they present two different splitting schemes employed to solve numerical problems. These schemes entail diverse approaches to manage the shape parameter and evaluate their influence on the model's effectiveness. In recent years, significant attempts have been made to develop reliable computational techniques to solve non-linear partial differential equations (PDEs) [19], which are prevalent in fluid mechanics and heat transfer. A renowned example of such equations is the Burgers' equation, famed for its combination of non-linear convection and diffusive impacts. As a non-linear PDE, it presents several intricate engineering hurdles. The presence of both non-linear convective terms $u(\frac{\partial u}{\partial x})$ and diffusive terms $\nu\frac{\partial^2 u}{\partial x^2}$ in Burgers' equation amplifies its complexity. This formula has practical use in several fields, such as fluid mechanics, compressible flows, shockwave theory, nonlinear acoustics, traffic flow, and turbulence phenomena. It has an influence on magneto-hydrodynamics, especially in systems like MHD steam plants and power generators that include conductive liquids within magnetic fluids [3]. Bonkile et al. [6] provide a detailed literature review of the Burgers' equation, discussing its physical and mathematical significance. They highlight persistent numerical challenges in achieving accuracy, stability, and convergence in various computational methods. Various methods have been proposed in the literature for solving the nonlinear Burgers' equation, each with its own advantages and disadvantages. Among the most recent and promising methods, we can refer to a non-exhaustive list, including the finite difference method [5, 15], finite volume method [14], finite element method [20, 26, 36], Implicit Exponential Finite-Difference Method [15], Method of lines [5, 18], Meshless method, Cubic Hermite Collocation Method [12], wavelet method [13], and Haar Wavelets and Finite Difference method [21, 27]. These methods tackle challenges regarding precision, reliability, and convergence. They have been implemented in various forms of the Burgers' equation, resulting in enhanced computational effectiveness and accuracy when solving this intricate Partial Differential Equation. Ashpazzadeh et al. [2] introduced a novel method for constructing wavelet bases on the interval [0, 1] derived from symmetric biorthogonal multiwavelets on the real line. The study utilises Hermite cubic spline multiwavelets on [0, 1] to solve the one-dimensional Burgers' equation through the application of the Mixed Finite Difference and Collocation Method (MFDCM). The suggested approach displays effectiveness and precision via numerical simulations, where the authors highlight its computational advantages, simplicity and sparsity. Nemati Saray et al. [28] introduce a Galerkin method using multiwavelets to efficiently solve the two-dimensional Burgers' equation. By utilizing the Crank-Nicolson scheme for time discretization, the resulting partial differential equations are converted into sparse systems of algebraic equations. The method's computational cost is dependent on the number of nonzero coefficients, and the findings suggest that error control can be achieved by adjusting the threshold accordingly. The Keller-Box scheme is a useful tool for addressing complex nonlinearities in various fields. It is particularly advantageous for solving parabolic partial differential equations numerically as it can handle robust nonlinearities that may prove challenging for traditional analytical approximations. As an implicit numerical approach, the method provides reliable and precise results while guaranteeing second-order accuracy in both spatial and temporal dimensions. Consequently, it is a powerful tool for handling complex nonlinear challenges across a range of domains. Prakash et al. [29] used a modified Keller-Box Scheme and Hopf-Cole transformation to present a numerical solution for the unsteady viscous Burgers' equation. Iqbal et al. [16] studied stagnation point flow over an electromagnetic surface using the Keller-Box Scheme. Vynnycky and Mitchell [35] use the Keller-Box finite-difference method to investigate the consequences of discontinuity in the boundary conditions when solving the linear one-dimensional transient heat equation. Their results show that despite its formal second order accuracy, this scheme can suffer from loss of accuracy. However, they show that a comprehensive understanding of the behaviour of the solution allows the development of a formulation that restores accuracy. In addition, they present benchmark calculations that provide valuable insights into the numerical solution of nonlinear parabolic PDEs that lack closed-form analytical solutions. The objective of this paper is to create a precise and reliable numerical resolution for the one-dimensional Burgers' equation, considering non-local boundary conditions. This will be achieved by utilising the finite difference technique, integrating the Keller-Box approach. To the best of our knowledge, this is the first time in the literature that the solution of the nonlinear Bergur's equation with nolocal boundary condition is addressed using the Keller-Box scheme to improve the accuracy and efficiency of the numerical results. The manuscript is organised as follows: in the next section, we present the governing equations. Then we present our numerical results. This is followed by a discussion in context. Finally, we conclude with a summary of the main results.

2. Governing Equation

The one-dimensional Burgers' equation for unsteady and viscous flow can be expressed as:

$$\mathrm{u_t} - \nu \mathrm{u_{xx}} = -\mathrm{u}\mathrm{u_x}, \quad for \quad 0 \leq \mathrm{x} \leq 1, \quad 0 \leq \mathrm{t}, \tag{2.1}$$

where u, x, v, and t represent the velocity, spatial coordinate, kinematic viscosity, and time respectively. Subject the Burgers' equation to the following initial condition:

$$u(x,0) = x. \tag{2.2}$$

with Dirichlet and nonlocal boundary conditions:

$$u(0,t) = 0, \tag{2.3}$$

$$\int_0^1 u(x,t)dx = \frac{1}{2(1+t)}. \tag{2.4}$$

To solve the Burgers' equation, we first convert the above nonlocal boundary equation above into a manageable local equation. This transformation is achieved by introducing the following equation:

$$\mathrm{u(x,t)} = \mathrm{v_x(x,t)}. \tag{2.5}$$

Burgers' equation (Eq. (2.1)), the initial condition (Eq. (2.2)), and the boundary conditions (Eq. (2.3) and Eq. (2.4)) thus becoming a modified system of equations:

$$v_{tx} - \nu v_{xxx} = -v_x v_{xx}, \quad for \quad 0 \leq x \leq 1, \quad t \geq 0, \tag{2.6}$$

$$u(x,0) = x, \tag{2.7}$$

$$u(0,t) = 0, \tag{2.8}$$

$$v(0,t) = 0, \tag{2.9}$$

$$v(1,t) = \frac{1}{2(1+t)}. \tag{2.10}$$

3. Finite Difference Scheme

To discretize the above modified equations (Eqs. (2.5)-(2.6)), we divide the solution domain into a uniform grid described by the set of nodes (x_j, t_n)defined below:

$$\mathrm{t_n} = \mathrm{n}\Delta\mathrm{t}, \quad \mathrm{n} = 0, 1, \ldots, \mathrm{N_{max}}, \tag{3.1}$$

$$\mathrm{x_j} = \mathrm{j}\Delta\mathrm{x}, \quad \mathrm{j} = 0, 1, \ldots, \mathrm{J_{max}}. \tag{3.2}$$

4. Keller-Box Method

By additional variable transformation, we simplify the partial differential equation (Eq. (2.6)) into first derivatives to implement the Keller-Box Method.

$$w(x,t) = u_x(x,t). \tag{4.1}$$

Substituting Eq. (2.5) and Eq. (4.1) in Eq. (2.6):

$$u_t - \nu w_x = -uw, \quad for \quad 0 \le x \le 1, \quad t \ge 0. \tag{4.2}$$

We consider four points around the node $(n+\frac{1}{2}, j-\frac{1}{2})$ as shown in (Figure 1) to develop the central discretization in space and time.

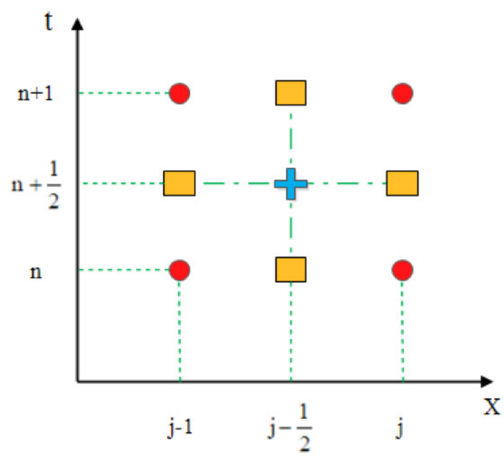


Figure 1. Schematic representation of domain for Keller-Box method.

Subsequently, the finite difference representation of the governing system of Equations (2.5), (4.1), and (4.2) is as follows:

$$\mathrm{u}_{j-\frac{1}{2}}^{\mathrm{n+1}} = \frac{\mathrm{v}_{\mathrm{j}}^{\mathrm{n+1}} - \mathrm{v}_{\mathrm{j-1}}^{\mathrm{n+1}}}{\Delta \mathrm{x}}, \tag{4.3}$$

$$\mathrm{w}_{\mathrm{j}-\frac{1}{2}}^{\mathrm{n+1}} = \frac{\mathrm{u}_{\mathrm{j}}^{\mathrm{n+1}} - \mathrm{u}_{\mathrm{j-1}}^{\mathrm{n+1}}}{\Delta \mathrm{x}}, \tag{4.4}$$

$$\frac{u_{j-\frac{1}{2}}^{n+1} - u_{j-\frac{1}{2}}^{n}}{\Delta t} + (uw)_{j-\frac{1}{2}}^{n+\frac{1}{2}} = v\frac{w_j^{n+\frac{1}{2}} - w_{j-1}^{n+\frac{1}{2}}}{\Delta x}. \tag{4.5}$$

In the above equations, the discretized terms containing the subscript $n+\frac{1}{2}$ or superscript $j-\frac{1}{2}$ index are approximated by the average values of the adjacent nodes:

$$u_{j-\frac{1}{2}}^{n+1} = \frac{u_j^{n+1} + u_{j-1}^{n+1}}{2}, \tag{4.6}$$

$$w_{j-1}^{n+\frac{1}{2}} = \frac{w_{j-1}^{n+1} + w_{j-1}^{n}}{2}, \tag{4.7}$$

$$(uw)_{j-\frac{1}{2}}^{n+\frac{1}{2}} = \frac{(uw)_j^{n+1} + (uw)_{j-1}^{n+1} + (uw)_j^n + (uw)_{j-1}^n}{4}. \tag{4.8}$$

Where (uw) $)_j^n$ is equal $u_j^n w_j^n$. Similar to Eqs. (4.6), (4.7), and (4.8), further terms $\left(w_{j-\frac{1}{2}}^{n+1}, u_{j-\frac{1}{2}}^{n}\right.$ and $\left.w_j^{n+\frac{1}{2}}\right)$ would be derived.

5. Linearization of the System of Equations

To enable the implementation of the Keller-Box method, we should first linearize the system of discretized equations (Eq. (4.5)) by imposing the following representations:

$$\left(w_j^{n+1}\right)^{k+1} = \left(w_j^{n+1}\right)^k + \left(\delta w_j^{n+1}\right)^k, \tag{5.1}$$

$$\left(u_{j-1}^{n+1}\right)^{k+1} = \left(u_{j-1}^{n+1}\right)^k + \left(\delta u_{j-1}^{n+1}\right)^k, \tag{5.2}$$

$$\left(v_j^{n+1}\right)^{k+1} = \left(v_j^{n+1}\right)^k + \left(\delta v_j^{n+1}\right)^k, \tag{5.3}$$

$$\left((uw)_j^{n+1}\right)^{k+1} = \left(u_j^{n+1}\right)^{k+1}\left(w_j^{n+1}\right)^{k+1} = \left(\left(u_j^{n+1}\right)^k + \left(\delta u_j^{n+1}\right)^k\right)\left(\left(w_j^{n+1}\right)^k + \left(\delta w_j^{n+1}\right)^k\right). \tag{5.4}$$

Where k represents the number of iterations in each time step. Note that in each iteration $\left(\mathrm{w}_{\mathrm{j}}^{\mathrm{n}+1}\right)^{\mathrm{k}}$, $\left(u_{j-1}^{n+1}\right)^k$, $\left(v_j^{n+1}\right)^k$, $\left(u_j^{n+1}\right)^k$, $\left(w_j^{n+1}\right)^k$ are known variables, while $\left(\delta w_j^{n+1}\right)^k$, $\left(\delta u_{j-1}^{n+1}\right)^k$, $\left(\delta v_j^{n+1}\right)^k$, $\left(\delta \mathrm{u}_{\mathrm{j}}^{\mathrm{n}+1}\right)^{\mathrm{k}}$, $\left(\delta \mathrm{w}_{\mathrm{j}}^{\mathrm{n}+1}\right)^{\mathrm{k}}$ are assumed to be unknown variables.

6. Modified System of Equations

Applying the approximation and linearization mentioned in the previous sections, a modified system of equations is obtained:

$$\frac{\left(u_j^{n+1}\right)^k + \left(\delta u_j^{n+1}\right)^k + \left(u_{j-1}^{n+1}\right)^k + \left(\delta u_{j-1}^{n+1}\right)^k}{2} = \frac{\left(v_j^{n+1}\right)^k + \left(\delta v_j^{n+1}\right)^k - \left(v_{j-1}^{n+1}\right)^k - \left(\delta v_{j-1}^{n+1}\right)^k}{\Delta x}, \tag{6.1}$$

$$\frac{\left(w_j^{n+1}\right)^k + \left(\delta w_j^{n+1}\right)^k + \left(w_{j-1}^{n+1}\right)^k + \left(\delta w_{j-1}^{n+1}\right)^k}{2} = \frac{\left(u_j^{n+1}\right)^k + \left(\delta u_j^{n+1}\right)^k - \left(u_{j-1}^{n+1}\right)^k - \left(\delta u_{j-1}^{n+1}\right)^k}{\Delta x}, \tag{6.2}$$

$$\begin{aligned}
&\frac{\left(u_j^{n+1}\right)^k + \left(\delta u_j^{n+1}\right)^k + \left(u_{j-1}^{n+1}\right)^k + \left(\delta u_{j-1}^{n+1}\right)^k}{2\Delta t} - \frac{\left(u_j^{n}\right)^k + \left(u_{j-1}^{n}\right)^k}{2\Delta t} + \\
&+ \frac{\left(u_j^{n}\right)^k\left(w_j^{n}\right)^k + \left(u_{j-1}^{n}\right)^k\left(w_{j-1}^{n}\right)^k + \left(u_j^{n+1}\right)^k\left(w_j^{n+1}\right)^k + \left(u_j^{n+1}\right)^k\left(\delta w_j^{n+1}\right)^k}{4} + \\
&+ \frac{\left(w_j^{n+1}\right)^k\left(\delta u_j^{n+1}\right)^k + \left(u_{j-1}^{n+1}\right)^k\left(w_{j-1}^{n+1}\right)^k + \left(u_{j-1}^{n+1}\right)^k\left(\delta w_{j-1}^{n+1}\right)^k + \left(w_{j-1}^{n+1}\right)^k\left(\delta u_{j-1}^{n+1}\right)^k}{4} \\
&= v\frac{\left(w_j^{n}\right)^k + \left(w_j^{n+1}\right)^k + \left(\delta w_j^{n+1}\right)^k - \left(w_{j-1}^{n}\right)^k - \left(w_{j-1}^{n+1}\right)^k - \left(\delta w_{j-1}^{n+1}\right)^k}{2\Delta x}.
\end{aligned} \tag{6.3}$$

The unknown variables $\delta \mathrm{u}, \delta \mathrm{v}, \delta \mathrm{w}$ approach zero at convergence. Therefore, it is acceptable to ignore the quadratic and higher-order terms of $\delta u, \delta v, \delta w$ in Eq. (5.4). The boundary condition for this modified system of equations is:

$$\delta \mathrm{u}_1 = 0, \tag{6.4}$$

$$\delta \mathrm{v}_1 = 0, \tag{6.5}$$

$$\delta \mathrm{v}_{\mathrm{J}_{\max}} = 0. \tag{6.6}$$

7. Keller-Box Matrix

Consequently, the modified system of equations results in a tridiagonal block system of the following format:

$$C_j \delta_{j-1} + A_j \delta_j + B_j \delta_{j+1} = R_j \tag{7.1}$$

This format of the system of equations forms the following Keller-Box Matrix:

$$\begin{bmatrix} \mathrm{A}_1 & \mathrm{B}_1 & 0 & 0 & \cdots & 0 & 0 \\ \mathrm{C}_2 & \mathrm{A}_2 & \mathrm{B}_2 & 0 & \cdots & 0 & 0 \\ 0 & & & & & & 0 \\ \vdots & & \ddots & \ddots & \ddots & & \vdots \\ 0 & & & & & & 0 \\ 0 & 0 & \cdots & 0 & \mathrm{C}_{\mathrm{J}\max-1} & \mathrm{A}_{\mathrm{J}\max-1} & \mathrm{B}_{\mathrm{J}\max-1} \\ 0 & 0 & \cdots & 0 & 0 & \mathrm{C}_{\mathrm{J}\max} & \mathrm{A}_{\mathrm{J}\max} \end{bmatrix} \begin{bmatrix} \delta_1 \\ \delta_2 \\ \vdots \\ \\ \delta_{\mathrm{J}\max-1} \\ \delta_{\mathrm{J}\max} \end{bmatrix} = \begin{bmatrix} \mathrm{R}_1 \\ \mathrm{R}_2 \\ \vdots \\ \\ \mathrm{R}_{\mathrm{J}\max-1} \\ \mathrm{R}_{\mathrm{J}\max} \end{bmatrix}, \tag{7.2}$$

Where $\mathrm{A}, \mathrm{B}, \mathrm{C}, \delta$ and R are as follows:

$$\mathrm{A}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{\Delta \mathrm{x}_2} & \frac{1}{2} \end{bmatrix}, \tag{7.3}$$

$$A_j = \begin{bmatrix} -\frac{1}{\Delta x_j} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2\Delta t_n} + \frac{\left(w_j^{n+1}\right)^k}{4} & -\frac{\nu}{2\Delta x_j} + \frac{\left(u_j^{n+1}\right)^k}{4} \\ 0 & \frac{1}{\Delta x_{j+1}} & \frac{1}{2} \end{bmatrix}, \quad for \quad j = 2, 3, \ldots, J\max - 1, \tag{7.4}$$

$$\mathrm{A}_{\mathrm{J}\max} = \begin{bmatrix} -\frac{1}{\Delta \mathrm{x}_{\mathrm{J}\max}} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2\Delta \mathrm{t}_\mathrm{n}} + \frac{\left(\mathrm{w}_{\mathrm{J}\max}^{\mathrm{n}+1}\right)^\mathrm{k}}{4} & -\frac{\nu}{2\Delta \mathrm{x}_{\mathrm{J}\max}} + \frac{\left(\mathrm{u}_{\mathrm{J}\max}^{\mathrm{n}+1}\right)^\mathrm{k}}{4} \\ 1 & 0 & 0 \end{bmatrix}, \tag{7.5}$$

$$\mathrm{B}_\mathrm{j} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -\frac{1}{\Delta \mathrm{x}_{\mathrm{j}+1}} & \frac{1}{2} \end{bmatrix}, \quad for \quad \mathrm{j} = 1, 2, \ldots, \mathrm{J}\max - 1, \tag{7.6}$$

$$\mathrm{C}_\mathrm{j} = \begin{bmatrix} \frac{1}{\Delta \mathrm{x}_\mathrm{j}} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2\Delta \mathrm{t}_\mathrm{n}} + \frac{\left(\mathrm{w}_{\mathrm{j}-1}^{\mathrm{n}+1}\right)^\mathrm{k}}{4} & -\frac{\nu}{2\Delta \mathrm{x}_\mathrm{j}} + \frac{\left(\mathrm{u}_{\mathrm{j}-1}^{\mathrm{n}+1}\right)^\mathrm{k}}{4} \\ 0 & 0 & 0 \end{bmatrix}, \quad for \quad \mathrm{j} = 2, 3, \ldots, \mathrm{J}\max, \tag{7.7}$$

$$\delta_j = \begin{bmatrix} \left(\delta v_j^{n+1}\right)^k \\ \left(\delta u_j^{n+1}\right)^k \\ \left(\delta w_j^{n+1}\right)^k \end{bmatrix}, \quad for \quad j = 1, 2, \ldots, J\max, \tag{7.8}$$

$$R_j = \begin{bmatrix} \mathrm{RHS}_1(j) \\ \mathrm{RHS}_3(j) \\ \mathrm{RHS}_2(j+1) \end{bmatrix}, \quad for \quad j = 2, 3, \ldots, J\max -1, \tag{7.9}$$

$$\mathrm{R}_1 = \begin{bmatrix} 0 \\ 0 \\ \mathrm{RHS}_2(\mathrm{j}=2) \end{bmatrix}, \tag{7.10}$$

$$\mathrm{R}_{\mathrm{J_{max}}} = \begin{bmatrix} \mathrm{RHS}_1(\,\mathrm{J}\max) \\ \mathrm{RHS}_3(\,\mathrm{J}\max) \\ 0 \end{bmatrix}. \tag{7.11}$$

The terms $\mathrm{RHS}_1, \mathrm{RHS}_2$ and RHS_3 for the $\mathrm{j^{th}}$ node are equivalent to:

$$\mathrm{RHS}_1(j) = \frac{\left(v_j^{n+1}\right)^k - \left(v_{j-1}^{n+1}\right)^k}{\Delta x} - \frac{\left(u_j^{n+1}\right)^k + \left(u_{j-1}^{n+1}\right)^k}{2}, \tag{7.12}$$

$$\mathrm{RHS}_2(j) = \frac{\left(u_j^{n+1}\right)^k - \left(u_{j-1}^{n+1}\right)^k}{\Delta x} - \frac{\left(w_j^{n+1}\right)^k + \left(w_{j-1}^{n+1}\right)^k}{2}, \tag{7.13}$$

$$\begin{aligned} \mathrm{RHS}_3(j) = {} & \frac{\left(u_j^{n}\right)^k + \left(u_{j-1}^{n}\right)^k - \left(u_j^{n+1}\right)^k - \left(u_{j-1}^{n+1}\right)^k}{2\Delta t} - \\ & - \frac{\left(u_j^{n}\right)^k \left(w_j^{n}\right)^k + \left(u_{j-1}^{n}\right)^k \left(w_{j-1}^{n}\right)^k + \left(u_j^{n+1}\right)^k \left(w_j^{n+1}\right)^k + \left(u_{j-1}^{n+1}\right)^k \left(w_{j-1}^{n+1}\right)^k}{4} + \\ & + \nu \frac{\left(w_j^{n}\right)^k + \left(w_j^{n+1}\right)^k - \left(w_{j-1}^{n}\right)^k - \left(w_{j-1}^{n+1}\right)^k}{2\Delta x}. \end{aligned} \tag{7.14}$$

8. Thomas Algorithm

To solve the tridiagonal block system of equations in the Keller-Box matrix (Eq. (7.2)), we utilized the Thomas algorithm. This algorithm is a kind of Gaussian elimination method where fewer operations are required to reach the solution. In this algorithm, the first step is to eliminate the blocks under the main block diagonal.

$$C_j^* = 0, \quad j = 2, 3, \ldots, J_{\max}, \tag{8.1}$$

$$A_j^* = A_j - \frac{C_j}{\left|A_{j-1}^*\right|} B_{j-1}, \quad j = 2, 3, \ldots, J_{\max}, \tag{8.2}$$

$$B_j^* = B_j, \quad j = 1, 3, \ldots, J_{\max} - 1, \tag{8.3}$$

$$A_1^* = A_1, \tag{8.4}$$

$$R_j^* = R_j - \frac{C_j}{\left|A_{j-1}^*\right|} R_{j-1}^*, \quad j = 3, 4, \ldots, J_{\max}, \tag{8.5}$$

$$R_2^* = R_2 - \frac{C_2}{\left|A_1^*\right|} R_1^*. \tag{8.6}$$

A backward substitution would then immediately lead to the unknown blocks.

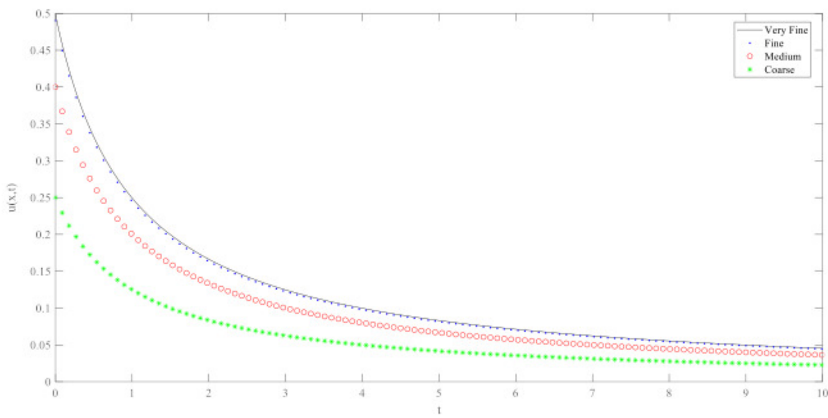
$$D^*_{J\max} = \text{inv}\left(A^*_{J_{\max}}\right) R^*_{J\max}\,, \tag{8.7}$$

$$D_j = \text{inv}\left(A^*_j\right)\left[R^*_j - B_j D^*_{j+1}\right], \quad j = J_{\max}-1,\ldots,1. \tag{8.8}$$

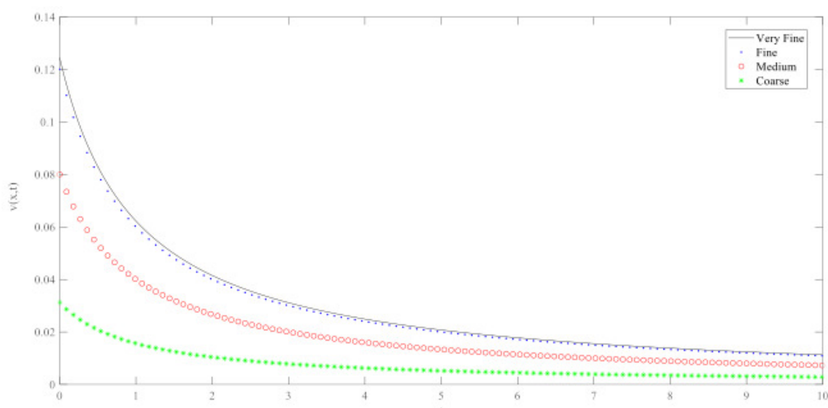
9. Results and discussion

In this section, the grid independency, validation and numerical results for Bergers' equation with Keller-Boxmethod are discussed.

10. Grid Independency Study



(a)



(b)

Figure 2. Plot of (a) $u(x,t)$ and (b) $v(x,t)$ versus time at $x = 0.5$ for different grid sizes (Very Fine: 1000 nodes, Fine: 100 nodes, Medium: 10 nodes, Course: 4 nodes).

Figure 2 (a) and (b) shows the resulting values for the unknown variables (u) and (v) as a function of the size of the grid structure. This figure shows that the resulting values for the unknown variables remain dependent on the size of the "Course" or "Medium" grid. However, the "Fine" grid approximately satisfies the independence of the grid from the above solution variables.

11. Validation

Numerical Results of this study are validated with the analytical solution of the Burgers' equation presented in [1]. Figure 3 sketches the numerical and analytical solution for the velocity in a fixed time (a) and in a fixed space (b). This figure indicates close agreement between the analytical solution and the numerical solution presented in this work.

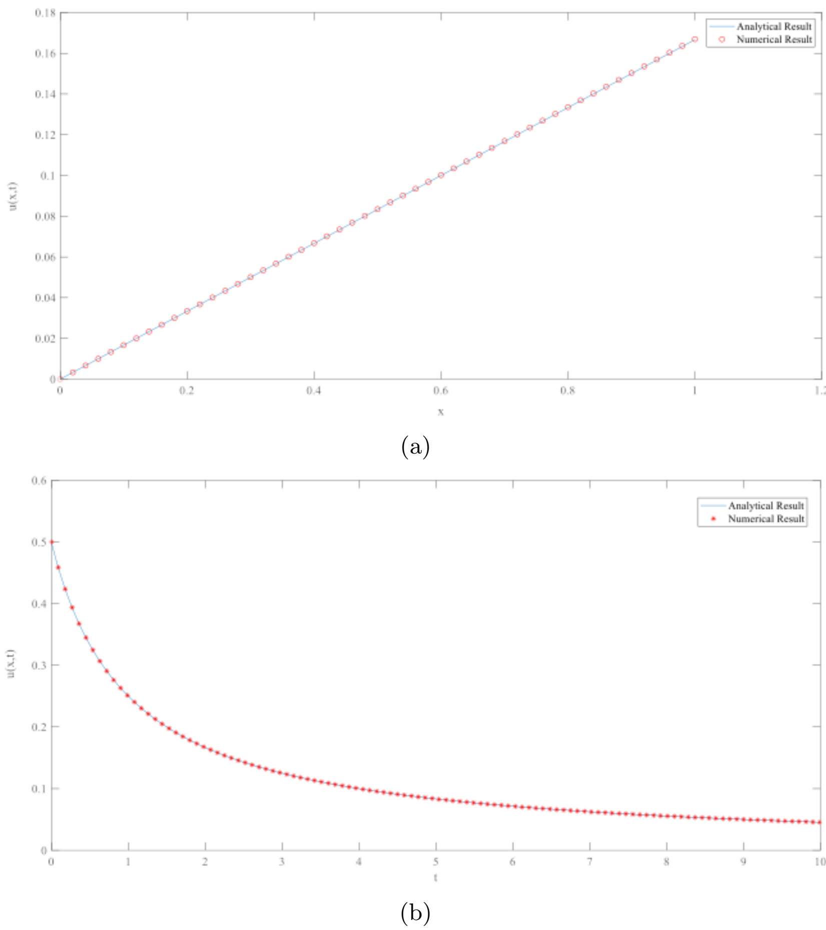


Figure 3. Plot of u(x, t) (a) at t = 5 during the space and (b) versus time at x = 0.5.

3.3 Numerical results. The numerical results were obtained after the development of a computer program to solve the Keller-Box matrix. The results indicate a numerical solution for Burgers' equation. As shown in Figure. 4, the velocity is linear in every time cross-section. On the other hand, if we consider a particular spatial coordinate, it is noticeable that the value of the velocity gradually and continuously decreases toward zero with time. In other words, the dependence on the boundary condition for the last node dissipates with time, while the first node is fixed to a single value due to the Dirichlet boundary condition. Consequently, the term (u_t) vanishes over infinite time. Consequently, the type of the partial differential equation (Eq. (4.2)) changes from parabolic to elliptic as time progresses to infinity. Figure 5 shows the behavior of the integral of velocity (v) in time and space. Similar to Figure 2.4, v(x, t) in this figure mimics the time evolution of velocity and transitions to a stationary value infinite time. Additionally, the time changes of the value of the last node are significantly higher than those of other positions in the spatial coordinate. However, in contrast to Figure 4, the integral of the velocity in space is parabolic at each time step (see Figure. 2.5).

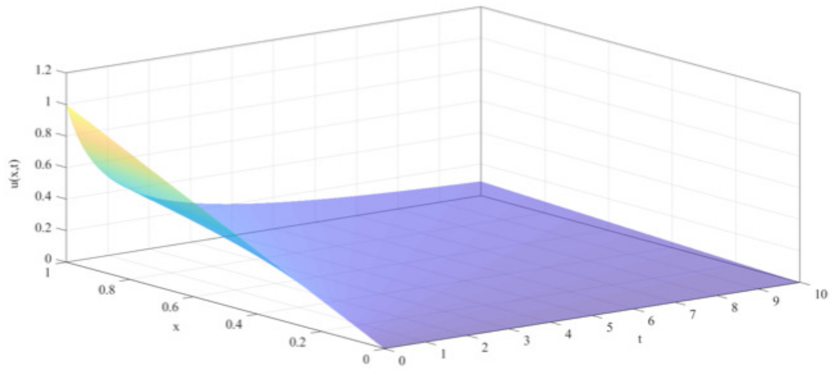


FIGURE 4. Surface representation of the velocity as a function of time and spatial coordinate.

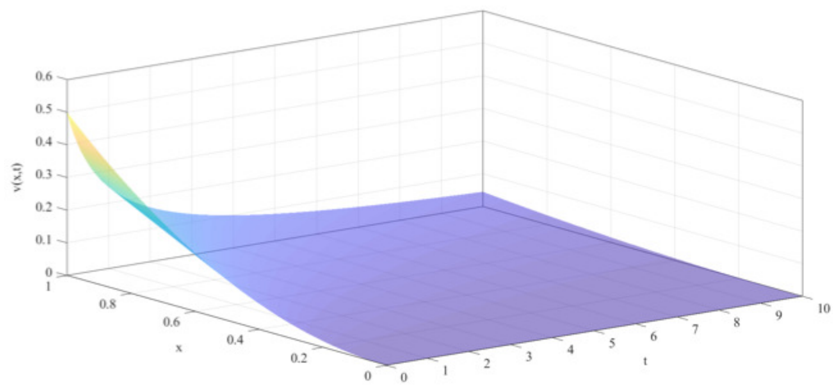


FIGURE 5. Surface representation of the integral of velocity as a function of time and spatial coordinate.

The derivative of velocity $w(x,t\ )$ is sketched in time and space (see Figure 6). At each time step, the value of $w(x,t)$ is constant over the entire range of space. Similar to the sketches for $u(x,t)$ and $v(x,t)$ in Figures 4 and 5, w(x,t) also reaches a plateau over time. The analytical solution of the Burgers' equation for the velocity is [1]:

$$u(x,t) = \frac{x}{1+t}. \tag{11.1}$$

Figure 7 displays the velocity error distribution over the entire range of the solution, with the convergence limit for the residuals sets to $1e-10$. The maximum difference between the analytical and the numerical solutions is about $4e-6$ and occurs at the last nodes of the first time steps. These points are very unstable in the first time steps, as can be seen in Figure 7. However, as time progresses, the errors radically disappear. A plausible explanation for this is that the values at these points undergo major transitions to reach the equilibrium state. Therefore, it takes time for the errors to disappear.

4 Conclusion

In this paper, the Burgers' Equation is considered as a typical nonlinear partial differential equation (PDE) for which a non-local boundary condition holds. By converting the non-local boundary condition into a local Boundary condition, we present a numerical solution using the Keller-Box Method and Thomas Algorithm to solve the non-linear PDE. The present numerical study has confirmed that the Keller-Box method is second-order spatially and

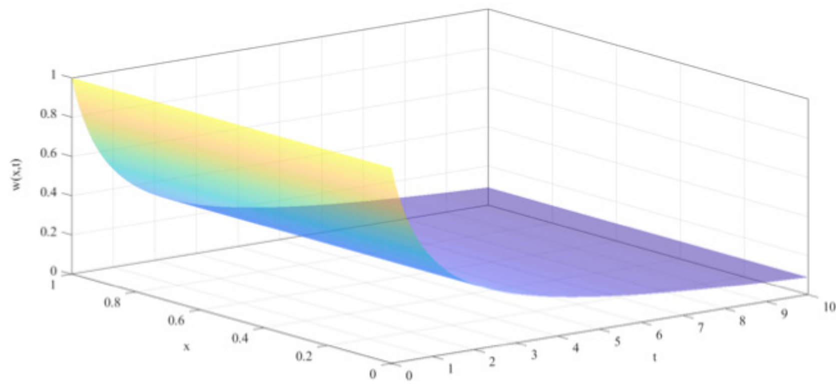


FIGURE 6. Surface representation of the derivative of the velocity as a function of time and spatial coordinate.

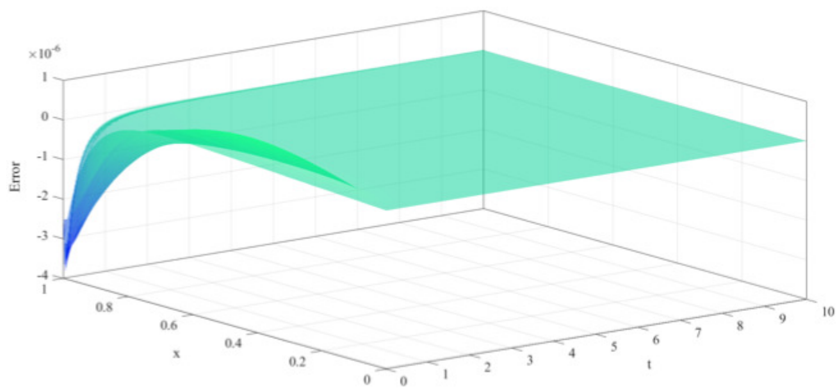


FIGURE 7. Surface representation of the velocity errors as the difference between the analytical and the numerical solutions as a function of time and spatial coordinates.

temporally. The analytical result of Burgers' equation is one of the rare non-linear equation solution obtained and developed in various science orientation problems, so it is important to find out the influences of various initials and boundary conditions on the behavior of Burgers' equation solutions. The results are quite convincing and are very close to the analytical solution. Future research could explore the application of our method to problems with shocks or discontinuities.

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REFERENCES

[1] N. Al-Zaid and H. Bakodah, *Iterative algorithm for parabolic and hyperbolic pdes with nonlocal boundary conditions*, Journal of Ocean Engineering and Science, *3*(4) (2018), 316–324.
[2] E. Ashpazzadeh, B. Han, and M. Lakestani, *Biorthogonal multiwavelets on the interval for numerical solutions of burgers' equation*, Journal of Computational and Applied Mathematics, *317* (2017), 510–534.
[3] J. Bec and K. Khanin, *Burgers turbulence*, Physics reports, *447*(1-2) (2007), 1–66.

[4] G. Berikelashvili and N. Khomeriki, *On a numerical solution of one nonlocal boundary-value problem with mixed dirichlet–neumann conditions*, Lithuanian Mathematical Journal, *53* (2013), 367–380.

[5] J. Biazar, Z. Ayati, and S. Shahbazi, *Solution of the burgers equation by the method of lines*, American Journal of Numerical Analysis, *2*(1) (2014) 1–3.

[6] M. P. Bonkile, A. Awasthi, C. Lakshmi, V. Mukundan, and V. Aswin, *A systematic literature review of burgers' equation with recent advances*, Pramana, *90* (2018), 1–21.

[7] A. Borhanifar, S. Shahmorad, and E. Feizi, *A matrix formulated algorithm for solving parabolic equations with nonlocal boundary conditions*, Numerical Algorithms, *74* (2017), 1203–1221.

[8] A. Bouziani, N. Merazga, and S. Benamira, *Galerkin method applied to a parabolic evolution problem with nonlocal boundary conditions*, Nonlinear Analysis: Theory, Methods & Applications, *69*(5-6) (2008), 1515–1524.

[9] J. R. Cannon, *The solution of the heat equation subject to the specification of energy*, Quarterly of Applied Mathematics, *2*(2) (1963), 155–160.

[10] M. Dehghan, *Numerical solution of a non-local boundary value problem with neumann's boundary conditions*, Communications in numerical methods in engineering, *19*(1) (2003), 1–12.

[11] M. Dehghan, *On the solution of an initial-boundary value problem that combines neumann and integral condition for the wave equation*, Numerical Methods for Partial Differential Equations: An International Journal, *21*(1) (2005), 24–40.

[12] I. A. Ganaie and V. Kukreja, *Numerical solution of burgers' equation by cubic hermite collocation method*, Applied Mathematics and Computation, *237* (2014), 571–581.

[13] K. Goyal and M. Mehra, *A fast adaptive diffusion wavelet method for burger's equation*, Computers & Mathematics with Applications, *68*(4) (2014), 568–577.

[14] Y. Guo, Y.-f. Shi, and Y.-m. Li, *A fifth-order finite volume weighted compact scheme for solving one-dimensional burgers' equation*, Applied Mathematics and Computation, *281* (2016) 172–185.

[15] B. Inan and A. R. Bahadir, *Numerical solution of the one-dimensional burgers' equation: Implicit and fully implicit exponential finite difference methods*, Pramana, *81* (2013), 547–556.

[16] Z. Iqbal, E. Azhar, Z. Mehmood, and E. Maraj, *Melting heat transport of nanofluidic problem over a riga plate with erratic thickness: Use of keller box scheme*, Results in physics, *7* (2017) 3648–3658.

[17] S. u. Islam, I. Aziz, and M. Ahmad, *Numerical solution of two-dimensional elliptic pdes with nonlocal boundary conditions*, Computers & Mathematics with Applications, *69*(3) (2015), 180–205.

[18] M. Javidi. *A numerical solution of burger's equation based on modified extended bdf scheme*, International Mathematical Forum, *1* (2006), pages 1565–1570.

[19] N. Jha and S. Verma, *Infinitely smooth multiquadric rbfs combined high-resolution compact discretization for nonlinear 2d elliptic pdes on a scattered grid network*, Computational Methods for Differential Equations, *11*(4) (2023), 753–775.

[20] R. Kannan and Z. Wang, *A high order spectral volume solution to the burgers' equation using the hopf–cole transformation*, International journal for numerical methods in fluids, *69*(4) (2012), 781–801.

[21] M. Kumar and S. Pandit, *A composite numerical scheme for the numerical simulation of coupled burgers' equation*, Computer Physics Communications, *185*(3) (2014) 809–817.

[22] Y. Lin, *Parabolic partial differential equations subject to nonlocal boundary conditions*, 1988. thesis (Ph.D.)–Washington State University.

[23] J. Martin-Vaquero and J. Vigo-Aguiar, *On the numerical solution of the heat conduction equations subject to nonlocal conditions*, Applied Numerical Mathematics, *59*(10) (2009), 2507–2514.

[24] S. Mesloub, *On a singular two dimensional nonlinear evolution equation with nonlocal conditions*, Nonlinear Analysis: Theory, methods & applications, *68*(9) (2008), 2594–2607.

[25] S. Mesloub and S. A. Messaoudi, *A nonlocal mixed semilinear problem for second-order hyperbolic equations* Electronic Journal of Differential Equations, *30* (2003), 1–17.

[26] R. Mittal and R. Jain, *Numerical solutions of nonlinear burgers' equation with modified cubic b-splines collocation method*, Applied Mathematics and Computation, *218*(15) (2012), 7839–7855.

[27] R. Mittal, H. Kaur, and V. Mishra, *Haar wavelet-based numerical investigation of coupled viscous burgers' equation* International Journal of Computer Mathematics, *92*(8) (2015), 1643–1659.

[28] B. Nemati Saray, M. Lakestani, and M. Dehghan, *On the sparse multiscale representation of 2-d burgers equations by an efficient algorithm based on multiwavelets*, Numerical Methods for Partial Differential Equations, *39*(3) (2023), 1938–1961.

[29] B. M. Prakash, A. Awasthi, and S. Jayaraj, *A numerical simulation based on modified keller box scheme for fluid flow: The unsteady viscous burgers' equation*, Mathematical Analysis and its Applications, Springer Proceedings in Mathematics & Statistics, *143* (2015), 565–575.

[30] L. S. Pulkina, *A nonlocal problem with integral conditions for a hyperbolic equation*, Differentsial'nye Uravneniya, *40*(7) (2004) 887–892.

[31] M. Ramezani, M. Dehghan, and M. Razzaghi, *Combined finite difference and spectral methods for the numerical solution of hyperbolic equation with an integral condition*, Numerical Methods for Partial Differential Equations: An International Journal, *24*(1) (2008), 1–8.

[32] S. Sajavičus, *Radial basis function method for a multidimensional linear elliptic equation with nonlocal boundary conditions*, Computers & Mathematics with Applications, *67*(7) (2014), 1407–1420.

[33] M. Siddique. *Smoothing of crank–nicolson scheme for the two-dimensional diffusion with an integral condition*, Applied mathematics and computation, *214*(2) (2009), 512–522.

[34] L. Ta-Tsien, *A class of non-local boundary value problems for partial differential equations and its applications in numerical analysis*, Journal of computational and applied mathematics, *28* (1989), 49–62.

[35] M. Vynnycky and S. Mitchell, *On the accuracy of a finite-difference method for parabolic partial differential equations with discontinuous boundary conditions*, Numerical Heat Transfer, Part B: Fundamentals, *64*(4) (2013) 275–292.

[36] G. Zhao, X. Yu, and R. Zhang, *The new numerical method for solving the system of two-dimensional burgers' equations*, Computers & Mathematics with Applications, *62*(8) (2011) 3279–3291, 2011.

[37] R. Zolfaghari and A. Shidfar, *Solving a parabolic pde with nonlocal boundary conditions using the sinc method*, Numerical Algorithms, *62* (2013), 411–427.