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An efficient algorithm for computing the eigenvalues of conformable Sturm-Liouville problem

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Abstract

In this paper, Computing the eigenvalues of the Conformable Sturm-Liouville Problem (CSLP) of order 2α, $\frac{1}{2} < \alpha \leq 1$, and dirichlet boundary conditions is considered. For this aim, CSLP is discretized to obtain a matrix eigenvalue problem (MEP) using finite element method with fractional shape functions. Then by a method based on the asymptotic form of the eigenvalues, we correct the eigenvalues of MEP to obtain efficient approximations for the eigenvalues of CSLP. Finally, some numerical examples to show the efficiency of the proposed method are given. Numerical results show that for the nth eigenvalue, the correction technique reduces the error order from $O(n^4h^2)$ to $O(n^2h^2)$.

Keywords. Sturm-Liouville problem, Conformable derivative, Finite elements method, Correction idea.
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1. Introduction

Sturm-Liouville problems are well-known one-dimensional eigenvalue problems. They play a fundamental role in both mathematics and physics. They appear for example in vibration, quantum mechanics and etc [15, 21, 33, 37]. Moreover, these problems appear in solving partial differential equations by separation of variables. Fractional version of Sturm-Liouville problems using several different types of fractional derivatives such as Riemann- Liouville, Caputo, and Conformable have been considered for research in recent years [16, 19, 22, 30, 32, 35, 40]. Most of the researchers try to develop the classical properties of the standard Sturm-Liouville problem to the fractional version.

Khalil [20] introduced the concept of conformable fractional derivative recently which maintains the properties of the standard derivative with integer order. The important properties of the conformable fractional derivative can be found in [1, 6, 20]. This version of the fractional derivative has applications in control theory, dynamical systems, quantum and Newton mechanics, calculus of variations, time scale problems and stochastic process [3, 7–9, 11, 14, 18, 24, 27, 31, 39]. A physical interpretation of the conformable derivative has been given by [38].
In this paper we consider a fractional Sturm-Liouville equation with Dirichlet boundary condition of the following form:

$$L_\alpha(q) := \begin{cases} D^\alpha D^\alpha y(x) + (\lambda - q(x))y(x) = 0, \\ y(0) = 0,\ y(\pi) = 0. \end{cases} \tag{1.1}$$

where D^α is the conformable fractional derivative of order $\frac{1}{2} < \alpha < 1$, λ is an eigenvalue, $y(x)$ is the corresponding eigenfunction and $q(x)$ is called potential function. The problem (1.1) is called *conformable Sturm-Liouville problem.*

It is proved that the problem (1.1) has infinitely many real and simple eigenvalues with no finite limit point. Moreover, the corresponding eigenfunctions make an orthogonal basis, see [30]. It is well-known that computing eigenvalues is impossible in general even for integer order Sturm-Liouville problems. Some spectral properties of CSLP and the integer case can be found in [2, 12, 16, 17, 30] Therefore, many researchers are trying to approximate

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the eigenvalues by using finite dimensional methods. Approximating the eigenvalues of the integer order Sturm-Liouville problems have been considered by many authors. Using finite dimensional methods such as finite difference [34], finite element [4], Numerov's method [5, 29] and other numerical methods [25, 26, 28] the eigenvalues of the Sturm-Liouville problem have been approximated. These methods discretize the given Sturm-Liouville problem to make a corresponding matrix eigenvalue problem (MEP) and approximate the eigenvalues of the given differential equation by the eigenvalues of MEP. As far as, we know this techniques also has been applied to Riemann-Liouville and Caputo types of fractional derivative [10, 13, 23, 36] but not for conformable fractional derivatives. We would like to apply this technique for problem (1.1) by using finite element method and adding a suitable correction term.

We organize the paper in the following manner. Some properties of the conformable derivatives and some related results concerning the problem (1.1) are given in section 2. We discretize the problem (1.1) to obtain the corresponding matrix eigenvalue problem in section 3. Moreover, we obtain the asymptotic form of the eigenvalues and introduce a method based on the asymptotic form to improve the eigenvalues of MEP. Finally, we present some numerical examples in the section 4 to illustrate the efficiency of the proposed method.

2. Preliminaries

This section gives a definition and some theorems of conformable fractional derivatives and conformable Sturm-Liouville's problem [1, 20, 30].

Definition 2.1. Suppose that f is a real function on $[0,\infty)$. Conformable integral and derivative of the function f of order α are as follows

$$D^\alpha f(x) = \lim_{h\to 0} \frac{f(x+hx^{1-\alpha}) - f(x)}{h}, \ \forall x>0, \ 0<\alpha\leq 1, \tag{2.1}$$

$$I^\alpha f(x) = \int_0^x f(s) d_\alpha s = \int_0^x s^{\alpha-1} f(s) ds, \ x>0. \tag{2.2}$$

Note that $D^\alpha f(0) = \lim_{x\to 0^+} D^\alpha f(x)$, and for differentiable function f at point x, we have $D^\alpha f(x) = x^{1-\alpha} f'(x)$. If $D^\alpha f(x)$ exists and is finite, we call that the function f is α-differentiable at point x.

Definition 2.2. For $n\in\mathbb{N}$, the space $C^n_\alpha[0,\pi]$ denotes all functions which are continuously α-differentiable up to order n.

Definition 2.3. For $p\geq 1$, the space $L^p_\alpha(0,\pi)$ denotes all functions satisfied in the condition $(\int_0^\pi |f(x)|^p d_\alpha x)^{\frac{1}{p}} < \infty$.

Theorem 2.4. *The following properties hold for eigenvalues and eigenfunctions of CSLP (1.1):*

i) All eigenvalues are simple and real,
ii) Eigenfunctions corresponding to different eigenvalues are orthogonal in $L^2_\alpha(0,\pi)$,
iii) The set of eigenfunctions of the problem $L_\alpha(q)$ is complete in $L^2_\alpha(0,\pi)$.

3. Matrix formulation and correction idea

In this section, using finite element method with fractional shape functions we transform the conformable Sturm-Liouville problem of the form (1.1) to a matrix eigenvalue problem. For this aim, we consider the standard division of the interval $[0,\pi]$ with interior points $x_i = ih; i=0,1,\ldots,N$, where $h=\frac{\pi}{N}$ is the step length of the division. We define the fractional shape functions as follows

$$\phi_{i,\alpha}(x) := \frac{1}{\alpha}\begin{cases} \frac{x^\alpha - x^\alpha_{i-1}}{x^\alpha_i - x^\alpha_{i-1}}, & x_{i-1}\leq x\leq x_i, \\ \frac{x^\alpha_{i+1} - x^\alpha}{x^\alpha_{i+1} - x^\alpha_i}, & x_i \leq x\leq x_{i+1}, \\ 0, & otherwise. \end{cases} \tag{3.1}$$

Using finite elements method with basis functions $\phi_{i,\alpha}(x)$, the problem (1.1) is transformed to the following matrix eigenvalue problem:

$$-Au + Fu = \Lambda Bu, \tag{3.2}$$

where A, B and F are tridiagonal matrices as follows:

$$\begin{cases} A_{ii} = -\frac{1}{\alpha h^{2\alpha}}\left[\frac{1}{i^\alpha-(i-1)^\alpha} + \frac{1}{(i+1)^\alpha - i^\alpha}\right], & i = 1, \dots, N, \\ A_{ii+1} = \frac{1}{\alpha h^{2\alpha}}\frac{1}{(i+1)^\alpha - i^\alpha}, & i = 1, \dots, N-1, \\ B_{ii} = \frac{1}{3\alpha^3}\left[(i+1)^\alpha - (i-1)^\alpha\right], & i = 1, \dots, N, \\ B_{ii+1} = \frac{1}{6\alpha^3}\left[(i+1)^\alpha - i^\alpha\right], & i = 1, \dots, N-1, \\ F_{ij} = \int_0^\pi q(x)\phi_i\phi_j \mathrm{d}_\alpha x, \quad i = j-1, j, j+1, & j = 1, 2, \dots, N. \end{cases}$$

Using this method the first N eigenvalues $\{\lambda_{n,\alpha}\}_{n=1}^N$ of the continuous problem (1.1) are approximated with eigenvalues $\{\Lambda_{n,\alpha}\}_{n=1}^N$ of the discrete system (3.2). In Tables 1-2, the errors $|\lambda_{n,\alpha} - \Lambda_{n,\alpha}|$ for different potential functions and different values of α are presented. To compare the results we compute the eigenvalues using the Matslise package [26]. The numerical results show that the error increases with respect to index n, significantly. Thus the eigenvalues of the matrix pair $(-A + F, B)$, are poor approximations. To improve the approximations, we explain and apply the correction technique for the eigenvalues of CSLP in the next subsection.

Table 1. Results for problem (1.1) with $q(x) = \exp(x^\alpha/\alpha)$, $N = 50$ and $\alpha = 0.6, 0.7, 0.8, 0.9$.

n	$\lvert\lambda_{n,0.6} - \Lambda_{n,0.6}\rvert$	$\lvert\lambda_{n,0.7} - \Lambda_{n,0.7}\rvert$	$\lvert\lambda_{n,0.8} - \Lambda_{n,0.8}\rvert$	$\lvert\lambda_{n,0.9} - \Lambda_{n,0.9}\rvert$
1	0.00823	0.00381	0.002538	0.00204
4	0.37934	0.17867	0.12138	0.10089
8	4.34073	2.38367	1.68808	1.45215
12	18.69715	11.58951	8.47075	7.34214
16	51.72182	35.13730	26.75324	23.39070
20	113.67560	82.27539	65.13835	57.61736
24	217.83388	163.93287	133.91116	120.05246
28	382.42327	292.68841	243.25387	221.03510
32	633.07654	484.33364	400.22993	366.44755
36	1006.32751	760.61489	608.90089	547.88676
40	1558.96148	1153.52616	877.19594	735.36652
44	2397.83470	1723.03085	1227.38412	906.01748
48	3839.11448	2657.70610	1755.23346	1093.11963

3.1. **Asymptotic form and corrected eigenvalues.** It was seen that for higher indices the eigenvalue $\Lambda_{n,\alpha}$ of (3.2) is a very poor approximation for the eigenvalue $\lambda_{n,\alpha}$ of problem (1.1). Thus by a method based on the asymptotic form of the eigenvalues we correct the eigenvalue $\Lambda_{n,\alpha}$ to obtain efficient approximation for the eigenvalue $\lambda_{n,\alpha}$. For this aim, we obtain the asymptotic form of the eigenvalues in the following theorem.

Theorem 3.1. *Let $\lambda_{n,\alpha}(q)$ and $\lambda_{n,\alpha}(0)$ be the eigenvalues of the problem (1.1) corresponding to $L_\alpha(q)$ and $L_\alpha(0)$, respectively. Then the following asymptotic formula as $n \to \infty$, holds:*

$$\lambda_{n,\alpha}(q) = \lambda_{n,\alpha}(0) + \frac{\alpha}{\pi^\alpha}\int_0^\pi q(x)d_\alpha x + c_n, \quad c_n = O(\frac{1}{n^2}). \tag{3.3}$$

TABLE 2. Results for problem (1.1) with $q(x) = 2 + \sin(\frac{\pi x^\alpha}{\alpha})$, $N = 50$ and $\alpha = 0.6, 0.7, 0.8, 0.9$.

n	$\lvert\lambda_{n,0.6} - \Lambda_{n,0.6}\rvert$	$\lvert\lambda_{n,0.7} - \Lambda_{n,0.7}\rvert$	$\lvert\lambda_{n,0.8} - \Lambda_{n,0.8}\rvert$	$\lvert\lambda_{n,0.9} - \Lambda_{n,0.9}\rvert$
1	0.00054	0.00048	0.00047	0.00049
4	0.20126	0.11993	0.09522	0.08706
8	3.80169	2.17641	1.60403	1.41834
12	17.81672	11.18970	8.29772	7.27546
16	50.55556	34.55826	26.48241	23.28525
20	112.25604	81.55298	64.78326	57.47560
24	216.14443	163.10676	133.50850	119.88878
28	380.37502	291.75663	242.86137	220.88254
32	630.52523	483.18083	399.88377	366.37158
36	1003.09398	758.97998	608.46079	547.99579
40	1554.795679	1151.02577	876.16653	735.61176
44	2391.95615	1719.14998	1224.92750	905.34818
48	3827.77624	2649.57861	1749.24395	1089.03939

Proof. Without loss of generality, we suppose that

$$\int_0^\pi q(x) d_\alpha x = 0. \tag{3.4}$$

We assume a solution for the problem (1.1) as follows

$$y(x) = A(x) \sin \varphi(x), \tag{3.5}$$

where $\varphi(0) = 0$ and $\varphi(\pi) = n\pi$. Also, we suppose that

$$D^\alpha y(x) = A(x)\sqrt{\lambda - q(x)} \cos \varphi(x). \tag{3.6}$$

Differentiating of (3.5), we find

$$D^\alpha y(x) = D^\alpha A(x) \sin \varphi(x) + D^\alpha \varphi(x) A(x) \cos \varphi(x). \tag{3.7}$$

Equating (3.6) and (3.7) we obtain

$$D^\alpha \varphi(x) = \sqrt{\lambda - q(x)} - \frac{D^\alpha A}{A} . \frac{\sin \varphi(x)}{\cos \varphi(x)}. \tag{3.8}$$

Differentiating of (3.6) then substituting in (1.1) and doing some calculations we get

$$\frac{D^\alpha A}{A} = \frac{D^\alpha q}{2(\lambda - q(x))} + D^\alpha \varphi(x) . \frac{\sin \varphi(x)}{\cos \varphi(x)} - \sqrt{\lambda - q(x)} . \frac{\sin \varphi(x)}{\cos \varphi(x)}. \tag{3.9}$$

Combining (3.8) and (3.9) and doing some computations we obtain

$$D^\alpha \varphi(x) = \sqrt{\lambda - q(x)} + \frac{1}{4} \frac{D^\alpha q}{\lambda - q(x)} \sin 2\varphi(x).$$

By $\alpha-$integration we have

$$\varphi(x) = \int_0^x \sqrt{\lambda - q(x)} d_\alpha x + \frac{1}{4} \int_0^x \frac{D^\alpha q}{\lambda - q(x)} \sin 2\varphi(x) d_\alpha x, \tag{3.10}$$

and for $|\lambda| \to \infty$,

$$\varphi(x) = \int_0^x \sqrt{\lambda - q(x)} d_\alpha x + O(\frac{1}{\lambda}). \tag{3.11}$$

Substituting (3.11) in the right hand side of (3.10) then taking $x = \pi$, we obtain

$$n\pi = \int_0^\pi \sqrt{\lambda - q(x)} d_\alpha x + \frac{1}{4}\int_0^\pi \frac{D^\alpha q}{\lambda - q(x)} \sin\left(2\int_0^x \sqrt{\lambda - q(x)}\right) d_\alpha x + O(\frac{1}{\lambda^2}). \tag{3.12}$$

For sufficiently large values of λ, we have

$$\begin{aligned}\int_0^\pi \sqrt{\lambda - q(x)} d_\alpha x &= \sqrt{\lambda}\int_0^\pi (1 - \frac{q}{\lambda})^{\frac{1}{2}} d_\alpha x \\ &= \sqrt{\lambda}\int_0^\pi \Big[1 - \frac{q(x)}{2\lambda} + \frac{1}{4\lambda^2} q^2(x) + \dots\Big] d_\alpha x \\ &= \sqrt{\lambda}\frac{\pi^\alpha}{\alpha} + O(\frac{1}{\lambda^{\frac{3}{2}}}). \end{aligned}\tag{3.13}$$

Also we have

$$\frac{D^\alpha q(x)}{\lambda - q(x)} = \frac{D^\alpha q(x)}{\lambda} + O(\frac{1}{\lambda^2}),$$

$$\sin\left(2\int_0^x \sqrt{\lambda - q(x)} d_\alpha x\right) = \sin 2\sqrt{\lambda}\frac{x^\alpha}{\alpha} + O(\frac{1}{\sqrt{\lambda}}).$$

Using these relations then integration by parts we find

$$\begin{aligned}&\frac{1}{4}\int_0^\pi \frac{D^\alpha q}{\lambda - q(x)} \sin\left(2\int_0^x \sqrt{\lambda - q(x)} d_\alpha x\right) d_\alpha x \\ &\quad= \frac{1}{4}\int_0^\pi \Big[\frac{D^\alpha q(x)}{\lambda} + O(\frac{1}{\lambda^2})\Big]\Big[\sin 2\sqrt{\lambda}\frac{x^\alpha}{\alpha} + O(\frac{1}{\sqrt{\lambda}})\Big] d_\alpha x \\ &\quad= \frac{1}{4}\int_0^\pi \frac{D^\alpha q(x)}{\lambda} \sin 2\sqrt{\lambda}\frac{x^\alpha}{\alpha} d_\alpha x + O(\frac{1}{\lambda^{\frac{3}{2}}}) \\ &\quad= -\frac{1}{8\lambda^{\frac{3}{2}}}\Big[D^\alpha q(\pi)\cos(2\sqrt{\lambda}\frac{\pi^\alpha}{\alpha}) - D^\alpha q(0)\Big] \\ &\qquad+ \frac{1}{8\lambda^{\frac{3}{2}}}\int_0^\pi D^\alpha D^\alpha q(x)\cos(2\sqrt{\lambda}\frac{x^\alpha}{\alpha}) d_\alpha x \\ &\quad= O(\frac{1}{\lambda^{\frac{3}{2}}}). \end{aligned}\tag{3.14}$$

(3.15)

Considering (3.13) and (3.14) in (3.12) we get

$$\sqrt{\lambda} = \alpha n \pi^{1-\alpha} + O(\frac{1}{\lambda^{\frac{3}{2}}}),$$

and by an iteration, we obtain

$$\sqrt{\lambda} = \alpha n \pi^{1-\alpha} + O(\frac{1}{n^3}). \tag{3.16}$$

Now, if $\overline{q} = \frac{\alpha}{\pi^\alpha}\int_0^\pi q(x) d_\alpha x \neq 0$, then problem (1.1) can be reduced to the following form

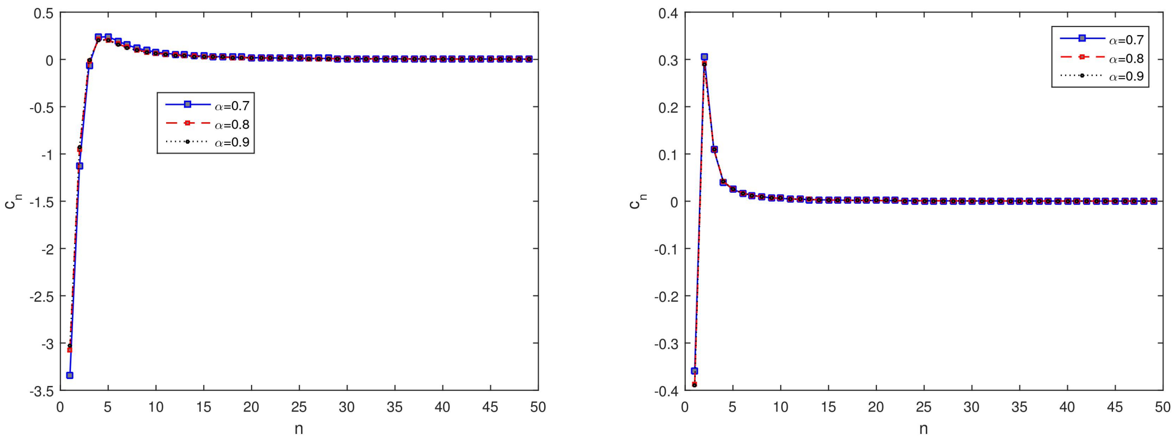
$$D^\alpha D^\alpha y + (\lambda^* - q^*(x))y = 0,$$

where $\lambda^* = \lambda - \overline{q}$, $q^*(x) = q(x) - \overline{q}$ and relation (3.16) hold for λ^*. Taking λ^* in (3.16) we obtain the following relation for nth eigenvalues

$$\lambda_{n,\alpha}(q) = \alpha^2\pi^{2-2\alpha}n^2 + \frac{\alpha}{\pi^\alpha}\int_0^\pi q(x) u d_\alpha x + O(\frac{1}{n^2}).$$

Since $\lambda_{n,\alpha}(0) = \alpha^2\pi^{2-2\alpha}n^2$, we obtain the asymptotic form (3.3). $\square$

In Figure 1 the values of c_n for different potential functions and different values of α are presented. The results of c_n satisfies the asymptotic form (3.3). According to the asymptotic formula (3.3) we observe that the increasing error



(a) $q_1(x) = \exp(x^\alpha/\alpha)$. (b) $q_2(x) = 2 + \sin(\pi x^\alpha/\alpha)$.

FIGURE 1. The values of constant c_n for different potential functions.

in approximation $\lambda_{n,\alpha}(q) \approx \Lambda_{n,\alpha}(q)$ is a result of an existing error in approximation $\lambda_{n,\alpha}(0) \approx \Lambda_{n,\alpha}(0)$. Therefore we expect the reduction of error by adding the correction term $\epsilon_{n,\alpha} = \lambda_{n,\alpha}(0) - \Lambda_{n,\alpha}(0)$ to the eigenvalues $\Lambda_{n,\alpha}(q)$. We have $\lambda_{n,\alpha}(0) = n^2\pi^{2-2\alpha}\alpha^2$ and the values of $\Lambda_{n,\alpha}(0)$ can be computed by using the function **eig**(A, B) in Matlab software. Thus we have the following corrected approximations:

$$\lambda_{n,\alpha} \approx \tilde{\Lambda}_{n,\alpha} := \Lambda_{n,\alpha} + \epsilon_{n,\alpha}.\ n = 1, 2, \ldots, N. \tag{3.17}$$

4. Numerical results

In this section, to show the efficiency of the proposed method for two potential functions and different values of α the eigenvalues of the problem (1.1) can be computed using corrected approximations (3.17).

Also,we investigate the rate of convergence for uncorrected and corrected numerical approximations. From the results of Tables 1 and 2, it was seen that the error in approximating the nth eigenvalue depends on index n and step size h. Thus we have an error order of the form $O(n^\gamma h^\beta)$.
The rate of convergence with respect to step size h and index n can be determined by using the following relations:

$$\beta := erc_\lambda^h(N, \alpha, n) = \log_2 \frac{\Lambda^N_{n,\alpha} - \Lambda^{\frac{N}{2}}_{n,\alpha}}{\Lambda^{2N}_{n,\alpha} - \Lambda^N_{n,\alpha}}, \qquad \gamma := erc_\lambda^n(N, \alpha, n) = \frac{\log \dfrac{\Lambda^N_{n,\alpha} - \Lambda^{\frac{N}{2}}_{n,\alpha}}{\Lambda^N_{n+1,\alpha} - \Lambda^{\frac{N}{2}}_{n+1,\alpha}}}{\log \frac{n}{n+1}}, \tag{4.1}$$

where erc_λ^h and erc_λ^n denote the rate of convergence with respect to step size h and index n, respectively. For computing the rate of convergence for corrected approximations (3.17), we replace $\Lambda_{n,\alpha}$ with $\tilde{\Lambda}_{n,\alpha}$ in the relaltions (4.1).

Example 4.1. We consider problem (1.1) with potential function $q(x) = e^{\frac{x^\alpha}{\alpha}}$ and different values of α. the error for the corrected approximations $\tilde{\Lambda}_{n,\alpha}$ are presented in Table 3. The rate of convergence for corrected and uncorrected approximations are given in Tables 5 and 6. It was seen that the correction idea reduces the error order from $O(n^4h^2)$ to $O(n^2h^2)$.

Table 3. Errors of corrected eigenvalues with $N = 50$, $q(x) = \exp(\frac{x^\alpha}{\alpha})$ and $\alpha = 0.6, 0.7, 0.8, 0.9$.

n	$\lvert\lambda_{n,0.6} - \tilde{\Lambda}_{n,0.6}\rvert$	$\lvert\lambda_{n,0.7} - \tilde{\Lambda}_{n,0.7}\rvert$	$\lvert\lambda_{n,0.8} - \tilde{\Lambda}_{n,0.8}\rvert$	$\lvert\lambda_{n,0.9} - \tilde{\Lambda}_{n,0.9}\rvert$
1	0.00785	0.00348	0.00221	0.00172
4	0.17197	0.05699	0.02554	0.01366
8	0.51684	0.20066	0.08165	0.03303
12	0.84543	0.38806	0.16847	0.06506
16	1.12264	0.56382	0.26450	0.10299
20	1.36708	0.70711	0.34845	0.13877
24	1.61438	0.81356	0.39891	0.16100
28	1.91814	0.91412	0.39739	0.15211
32	2.33624	1.08858	0.36350	0.08256
36	2.97286	1.44507	0.43204	0.08228
40	4.06005	2.18941	0.84311	0.19840
44	6.16494	3.85731	2.11338	0.44282
48	11.34145	8.53845	6.48151	4.29275

Example 4.2. We consider problem (1.1) with potential function $q(x) = 2 + \sin(\frac{\pi x^\alpha}{\alpha})$ and different values of α. the error for the corrected approximations $\tilde{\Lambda}_{n,\alpha}$ are presented in Table 4. The rate of convergence for corrected and uncorrected approximations are given in Tables 7 and 8. It was seen that the correction idea reduces the error order from $O(n^4h^2)$ to $O(n^2h^2)$.

Table 4. Errors of corrected eigenvalues with $N = 50$, $q(x) = 2 + \sin(\frac{\pi x^\alpha}{\alpha})$ and $\alpha = 0.6, 0.7, 0.8, 0.9$.

n	$\lvert\lambda_{n,0.6} - \tilde{\Lambda}_{n,0.6}\rvert$	$\lvert\lambda_{n,0.7} - \tilde{\Lambda}_{n,0.7}\rvert$	$\lvert\lambda_{n,0.8} - \tilde{\Lambda}_{n,0.8}\rvert$	$\lvert\lambda_{n,0.9} - \tilde{\Lambda}_{n,\ 0.9}\rvert$
1	0.00015	0.00015	0.00015	0.00016
4	0.00610	0.00175	0.00062	0.00017
8	0.02218	0.00659	0.00240	0.00078
12	0.03499	0.01174	0.00454	0.00161
16	0.04362	0.01521	0.00631	0.00245
20	0.05247	0.01530	0.00662	0.00298
24	0.07507	0.01254	0.00375	0.00267
28	0.13009	0.01764	0.00489	0.00044
32	0.21506	0.06421	0.01734	0.00660
36	0.26066	0.18984	0.00806	0.02674
40	0.10575	0.31098	0.18629	0.04683
44	0.28639	0.02356	0.34323	0.22648
48	0.00321	0.41096	0.49201	0.21250

TABLE 5. Scaled Errors of Λ_i and $\tilde{\Lambda}_i$ for $q(x) = \exp(\frac{x^\alpha}{\alpha})$.

k	N	$\alpha = 0.7$		$\alpha = 0.8$		$\alpha = 0.9$	
		erc^h_Λ	erc^n_Λ	erc^h_Λ	erc^n_Λ	erc^h_Λ	erc^n_Λ
1	50	–	–	–	–	–	–
	100	2.013	2.6656	2.001	2.613	1.999	2.590
	200	2.005	2.664	2.000	2.614	1.999	2.591
	400	2.002	2.663	2.000	2.614	1.999	2.591
	800	–	2.662	–	2.614	–	2.591
4	50	–	–	–	–	–	–
	100	2.013	3.546	2.002	3.638	2.000	3.711
	200	2.009	3.556	2.000	3.632	2.000	3.705
	400	2.004	3.553	2.000	3.631	2.000	3.703
	800	–	3.550	–	3.630	–	3.703
8	50	–	–	–	–	–	–
	100	1.981	3.883	2.008	3.962	2.009	3.984
	200	2.007	3.971	2.002	3.956	2.002	3.969
	400	2.007	3.992	2.000	3.954	2.000	3.964
	800	–	3.991	–	3.953	–	3.963
12	50	–	–	–	–	–	–
	100	1.916	3.831	2.011	3.996	2.020	4.027
	200	1.984	3.970	2.004	3.998	2.005	4.000
	400	2.005	4.048	2.001	3.996	2.001	3.997
	800	–	4.063	–	3.996	–	3.990
16	50	–	–	–	–	–	–
	100	1.859	3.782	2.008	3.990	2.033	4.047
	200	1.943	3.904	2.004	3.006	2.009	4.014
	400	1.995	4.034	2.001	3.009	2.024	4.000
	800	–	4.080	–	3.008	–	3.997
20	50	–	–	–	–	–	–
	100	1.831	3.767	2.003	3.978	2.041	4.049
	200	1.909	3.869	2.003	4.005	2.013	4.019
	400	1.981	4.016	2.001	4.011	2.003	4.003
	800	–	4.079	–	4.011	–	4.000

Table 6. Scaled Errors of Λ_i and $\tilde{\Lambda}_i$ for $q(x) = \exp(\frac{x^\alpha}{\alpha})$.

k	N	$\alpha = 0.7$		$\alpha = 0.8$		$\alpha = 0.9$	
		$erc^h_{\tilde{\Lambda}}$	$erc^n_{\tilde{\Lambda}}$	$erc^h_{\tilde{\Lambda}}$	$erc^n_{\tilde{\Lambda}}$	$erc^h_{\tilde{\Lambda}}$	$erc^n_{\tilde{\Lambda}}$
1	50	–	–	–	–	–	–
	100	2.014	2.348	2.001	2.160	2.000	2.019
	200	2.006	2.346	2.000	2.161	2.000	2.021
	400	2.002	2.343	2.000	2.161	2.000	2.021
	800	–	2.341	–	2.161	–	2.021
4	50	–	–	–	–	–	–
	100	2.005	1.712	1.992	1.462	1.983	0.916
	200	2.013	1.781	1.998	1.484	1.998	0.941
	400	2.007	1.784	2.000	1.489	1.995	0.949
	800	–	1.780	–	1.489	–	0.951
8	50	–	–	–	–	–	–
	100	1.841	1.633	1.941	1.793	1.936	1.626
	200	1.983	2.012	1.988	1.933	1.983	1.725
	400	2.008	2.124	1.997	1.962	1.995	1.749
	800	–	2.139	–	1.969	–	1.755
12	50	–	–	–	–	–	–
	100	1.568	1.186	1.818	1.556	1.853	1.581
	200	1.881	1.781	1.961	1.915	1.963	1.826
	400	1.990	2.077	1.991	2.000	1.990	1.882
	800	–	2.154	–	2.019	–	1.895
16	50	–	–	–	–	–	–
	100	1.305	0.800	1.628	1.150	1.716	1.297
	200	1.734	1.504	1.914	1.818	1.934	1.815
	400	1.944	1.953	1.981	1.988	1.983	1.917
	800	–	2.127	–	2.027	–	1.942
20	50	–	–	–	–	–	–
	100	1.122	0.604	1.437	0.873	1.562	1.052
	200	1.617	1.379	1.865	1.751	1.905	1.790
	400	1.895	1.876	1.970	1.973	1.976	1.922
	800	–	2.104	–	2.025	–	1.954

TABLE 7. Scaled Errors of Λ_i and $\tilde{\Lambda}_i$ for $q(x) = 2 + \sin(\frac{\pi x^\alpha}{\alpha})$.

k	N	$\alpha = 0.7$		$\alpha = 0.8$		$\alpha = 0.9$	
		erc^h_Λ	erc^n_Λ	erc^h_Λ	erc^n_Λ	erc^h_Λ	erc^n_Λ
1	50	–	–	–	–	–	–
	100	1.999	3.679	2.000	3.551	1.999	3.465
	200	2.000	3.670	2.000	3.548	2.000	3.464
	400	2.000	3.667	2.000	3.548	2.000	3.463
	800	–	3.666	–	3.548	–	3.364
4	50	–	–	–	–	–	–
	100	2.0178	4.223	2.005	4.085	2.002	4.027
	200	2.008	4.223	2.001	4.079	2.000	4.022
	400	2.003	4.216	2.000	4.078	2.000	4.021
	800	–	4.212	–	4.076	–	4.021
8	50	–	–	–	–	–	–
	100	1.994	4.064	2.012	4.061	2.010	4.032
	200	2.010	4.149	2.003	4.054	2.002	4.017
	400	2.007	4.167	2.000	4.051	2.000	4.013
	800	–	4.165	–	4.050	–	4.012
12	50	–	–	–	–	–	–
	100	1.929	3.910	2.015	4.040	2.022	4.047
	200	1.988	4.055	2.005	4.426	2.001	4.019
	400	2.006	4.131	2.001	4.040	2.005	4.011
	800	–	4.145	–	4.039	–	4.009
16	50	–	–	–	–	–	–
	100	1.869	3.822	2.012	4.015	2.034	4.058
	200	1.948	3.952	2.005	4.031	2.01	4.025
	400	1.996	3.083	2.001	4.034	2.005	4.011
	800	–	4.128	–	4.033	–	4.008
20	50	–	–	–	–	–	–
	100	1.839	3.797	2.007	4.015	2.042	4.057
	200	1.913	3.906	2.004	4.025	2.013	4.028
	400	1.983	4.055	2.001	4.031	2.003	4.012
	800	–	4.1118	–	4.031	–	4.007

TABLE 8. Scaled Errors of Λ_i and $\tilde{\Lambda}_i$ for $q(x) = 2 + \sin(\frac{\pi x^\alpha}{\alpha})$.

k	N	$\alpha = 0.7$		$\alpha = 0.8$		$\alpha = 0.9$	
		$erc^h_{\tilde{\Lambda}}$	$erc^n_{\tilde{\Lambda}}$	$erc^h_{\tilde{\Lambda}}$	$erc^n_{\tilde{\Lambda}}$	$erc^h_{\tilde{\Lambda}}$	$erc^n_{\tilde{\Lambda}}$
1	50	–	–	–	–	–	–
	100	1.991	0.857	1.997	2.327	1.997	1.210
	200	1.997	0.833	1.999	2.331	2.009	1.204
	400	1.998	0.829	2.000	2.331	2.001	1.202
	800	–	0.826	–	2.327	1.982	1.208
4	50	–	–	–	–	–	–
	100	1.984	2.135	1.966	2.109	1.982	2.378
	200	1.989	2.185	1.990	2.168	1.996	2.423
	400	1.994	2.200	1.996	2.185	1.994	2.433
	800	–	2.208	–	2.188	–	2.4301
8	50	–	–	–	–	–	–
	100	1.869	1.481	1.857	1.608	1.904	1.828
	200	1.961	1.778	1.961	1.848	1.978	1.998
	400	1.982	1.834	1.989	1.907	1.993	2.035
	800	–	1.856	–	1.928	–	2.043
12	50	–	–	–	–	–	–
	100	1.608	0.856	1.654	1.137	1.760	1.461
	200	1.918	1.689	1.915	1.749	1.948	1.887
	400	1.968	1.785	1.977	1.869	1.987	1.968
	800	–	1.810	–	1.900	–	1.986
16	50	–	–	–	–	–	–
	100	1.144	0.361	1.293	0.087	1.517	0.815
	200	1.853	1.516	1.854	1.641	1.906	1.791
	400	1.959	1.771	1.961	1.845	1.977	1.937
	800	–	1.782	–	1.895	–	1.969
20	50	–	–	–	–	–	–
	100	0.557	1.538	0.780	1.125	1.213	0.212
	200	1.776	1.408	1.797	1.573	1.865	1.736
	400	1.957	1.776	1.947	1.833	1.968	1.923
	800	–	1.782	–	1.894	–	1.965

5. Conclusion

In this paper, we applied the idea of the correction technique to compute the eigenvalues of conformable Sturm-Liouville problem. We discretized the Sturm-Liouville equation using finite element method with a fractional basis function to obtain a matrix eigenvalue problem. By adding a suitable correction term to the eigenvalues of the obtained matrix we find efficient approximations for the eigenvalues of FSLP (1.1). Numerical results show that, by using the correction term the error is reduced significantly. The rate of convergence is computed and the results show that the convergence rate reduces from 4 to 2 with respect to index n.

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