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Lie symmetry analysis for computing invariant manifolds associated with equilibrium solutions

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Abstract

We present a novel computational approach for computing invariant manifolds that correspond to equilibrium solutions of nonlinear parabolic partial differential equations (or PDEs). Our computational method combines Lie symmetry analysis with the parameterization method. The equilibrium solutions of PDEs and the solutions of eigenvalue problems are exactly obtained. As the linearization of the studied nonlinear PDEs at equilibrium solutions yields zero eigenvalues, these solutions are non-hyperbolic, and some invariant manifolds are center manifolds. We use the parameterization method to model the infinitesimal invariance equations that parameterize the invariant manifolds. We utilize Lie symmetry analysis to solve the invariance equations. We apply our framework to investigate the Fisher equation and the Brain Tumor growth differential equation.

Keywords. Lie symmetry analysis, Parameterization method, Equilibrium solution, Eigenvalue problem, Invariant manifolds, Invariance equation, Tanh method.

2010 Mathematics Subject Classification. 22E70, 35K57, 37D10.

1. Introduction

The focus of this paper is to investigate the geometric theory of differential equations. The geometry of these differential equations is determined by obtaining the equilibrium solutions, collecting the solutions as invariant manifolds, and analyzing the stability or instability of these solutions. Equilibria and periodic orbits and their corresponding asymptotic solutions in forward and backward time can explain the global analysis of a dynamical system. The invariant objects that correspond to these solutions provide a framework for understanding the dynamics of the phase space. We compute the non-trivial and non-hyperbolic equilibrium solutions of the differential equations and analyze the stability of the dynamical systems. We utilize Lie symmetry group analysis for gaining invariant manifolds associated with equilibrium solutions of evolution equations. We concentrate on two evolution equations whose general form is

$$\frac{\partial}{\partial t}Q = EQ + S(Q)\cdot \tag{1.1}$$

Where E is an elliptic operator and S is a semilinear operator. For instance, we aim to investigate the Fisher equation and the Brain Tumor growth differential equation, which are both instances of Equation (1.1). A reaction-diffusion equation was introduced as a dynamical system by Fisher for the first time and was called the Fisher equation [9]. Fisher equation has numerous applications and is a powerful tool for analysing many problems in fluid mechanics, chemicals, plasma physics, and other fields. The parameterization method is utilized to compute invariant objects.

We compute the equilibrium solutions of the Fisher equation, Brain Tumor growth differential equation, and the invariant manifolds attached to them by the Lie symmetry analysis group and tanh method. While numerical approaches are commonly used to compute invariant objects, we aim to utilize Lie symmetry group analysis instead. For instance, researchers have developed computer-assisted proofs for the existence of invariant objects in semilinear PDEs within a theoretical framework [8]. Periodic orbits, equilibrium points, traveling waves, and invariant manifolds attached to fixed points or periodic orbits are the invariant objects considered in [8]. The Chebyshev-Taylor spectral

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method was developed [16] for studying stable (or unstable) manifolds attached to periodic solutions of differential equations. High-order expansions of the chart maps for local finite-dimensional unstable manifolds of hyperbolic equilibrium solutions of scalar parabolic partial differential equations were studied [18]. Different approaches for computing a global stable (unstable) manifold of a vector field were surveyed [15]. A multiple shooting parameterization method for studying stable (unstable) manifolds attached to periodic orbits of systems, whose dynamics are determined by an implicit rule, was developed [19]. An efficient numerical method for computing Fourier–Taylor expansions of stable (unstable) manifolds associated with hyperbolic periodic orbits was presented [5].

Sophus Lie (1842-1899), a Norwegian mathematician, introduced many fundamental ideas for symmetry method. The local groups of point transformations of differential equations are groups of diffeomorphisms in the space of independent and dependent variables that map solutions of the differential equations into the other. These groups are referred to as the Lie symmetry groups of the differential equations. For further reading on the applications of Lie symmetry groups to differential equations, we recommend [14, 17]. Indeed, this method is the most effective method to gain exact solutions. Similarity variables are used to obtain the reduction equations. The similarity solutions for nonlinear PDEs are obtained by solving the associated reduction equations.

In section 2, we have reviewed the parameterization method for invariant manifolds of evolution problems on functional spaces. Section 3 presents the equilibrium solutions of the Fisher equation. We solve the eigenvalue equation and the invariance equation. To illustrate the results, we show some of the graphs of the manifolds. Section 4 is devoted to computing the equilibrium solutions of the Brain Tumor growth differential equation. We obtained the eigenvalues and eigenfunctions. The invariance equation has been modeled and solved. We parameterized the invariant manifolds by solving the invariance equation. Some invariant manifolds are displayed in graphs.

2. The parameterization method for the invariant manifold of vector fields

The parameterization method is a functional analytic approach for studying invariant manifolds [1–4, 6, 7, 10–12], and it is extensively studied in [13].

This section provides a review of the parameterization method used to compute invariant manifolds of evolution equations on a functional space. Assume that we are working with a functional space $\mathcal{F}$. Consider the following evolution equation

$$\frac{\partial}{\partial t}\varphi(t) = H(\varphi(t)), \tag{2.1}$$

where H is a Fréchet differentiable mapping. A solution curve (also called an orbit or trajectory) of the dynamical system (2.1) is a smooth curve $\phi : (\alpha, \beta) \longrightarrow \mathcal{F}$ that satisfies Eq. (2.1). If $(\alpha, \beta) = (0, \infty)$ ($(\alpha, \beta) = (-\infty, 0)$), then ϕ is the full forward orbit (the full backward orbit) of the dynamical system (2.1). Equilibrium solutions, which do not depend on time, are one of the important solution curves of a dynamical system. After getting the equilibrium solutions, we seek the linear and nonlinear stability of the equilibrium solutions. We analyze the solution curves in a neighborhood of the equilibrium solutions.

We consider $\vartheta\Delta\varphi(t, x) + h(\varphi(t, x), x)$ as the vector field H in Eq. (2.1), where $\vartheta > 0$. Therefore, our favourite equations are of the form

$$\frac{\partial}{\partial t}\varphi(t, x) = \vartheta\Delta\varphi(t, x) + h(\varphi(t, x), x)\cdot \tag{2.2}$$

Where $h : \mathbb{R} \times A \longrightarrow \mathbb{R}$ is a smooth function and A is a subset of $\mathbb{R}$. Therefore, solutions of the nonlinear elliptic equation

$$\vartheta\Delta\varphi(x) + h(\varphi(x), x) = 0, \tag{2.3}$$

are equilibrium solutions of the dynamical system (2.2). The pairs of eigenvalues and eigenfunctions $(\lambda, \mu(x))$ are obtained by solving the following eigenvalue problem

$$\vartheta\Delta(\mu(x)) + \partial_1 h(\varphi_0(x), x)\mu(x) = \lambda\mu(x), \tag{2.4}$$

where $\varphi_0(x)$ is an equilibrium solution. We assume the invariant manifolds parameterize by $\Phi : [r, s] \longrightarrow \mathcal{F}$ such that $0 \notin [r, s]$. To parameterize the invariant manifolds, we must solve the following invariance equation

$$\lambda\theta \frac{d}{d\theta}\Phi(\theta) = H(\Phi(\theta)), \tag{2.5}$$

where $\theta \in [r, s]$. Let $\Phi(\theta)(x) = \Phi(\theta, x)$, then the invariance equation ((2.5)) is reduced to

$$\lambda\theta \frac{\partial\Phi(\theta, x)}{\partial\theta} = \vartheta \frac{\partial^2\Phi(\theta, x)}{\partial x^2} + f(\Phi(\theta, x), x)\cdot \tag{2.6}$$

3. Parameterizing invariant manifolds of Fisher equation

The first evolution equation that we study in this paper is the Fisher equation. The one-dimensional Fisher equation is

$$\frac{\partial\varphi}{\partial t} = \frac{\partial^2\varphi}{\partial x^2} + \beta\varphi(1-\varphi)\cdot$$

3.1. **Computing equilibrium solutions, eigenvalues, and eigenfunctions.** In the first step to gain invariant manifolds, we solve equilibrium solutions. To find the equilibrium solutions for the Fisher equation, we need to solve the following nonlinear elliptic equation

$$\Delta_1 : \quad \frac{d^2\varphi}{dx^2} + \beta\varphi(1-\varphi) = 0, \quad \beta > 0\cdot \tag{3.1}$$

Consider the following one-parameter Lie group of infinitesimal transformations that act on the independent variable x and the dependent variable φ of Equation ((3.1))

$$\begin{aligned} \tilde{x} &= x + \sigma\xi(x, \varphi) + O(\sigma^2), \\ \tilde{\varphi} &= \varphi + \sigma\zeta(x, \varphi) + O(\sigma^2)\cdot \end{aligned}$$

The Lie symmetry is generated by the vector field

$$W = \xi(x, \varphi)\frac{\partial}{\partial x} + \zeta(x, \varphi)\frac{\partial}{\partial\varphi}\cdot$$

In the given Lie group, σ is a group parameter, while ξ and ζ are the infinitesimals for the independent variable x and the dependent variable φ, respectively. The functions ξ and ζ are the unknown functions that will be obtained by satisfying the following invariance condition

$$Pr^{(2)}W(\Delta_1)|_{\Delta_1=0} = 0\cdot \tag{3.2}$$

Here, $Pr^{(2)}W$ refers to the second order prolongation of W.
A system of PDEs is obtained by Extracting coefficients of φ_x and $\varphi_{x,x}$ of Equation (3.2). By solving the obtained system of PDEs, we can obtain the infinitesimal generators

$$\{\zeta(x, \varphi) = 0, \xi(x, \varphi) = A_1\},$$

such that A_1 is an arbitrary constant. Therefore, the Lie algebra $\langle\frac{\partial}{\partial x}\rangle$ is infinitesimal symmetries of Equation ((3.1)). We reverse the roles of the independent and dependent variables. We make the substitutions $y = \varphi$ and $v = x$. Therefore, Equation ((3.1)) is reduced to

$$\frac{-\frac{d^2}{dy^2}v(y)}{(\frac{d}{dy}v(y))^3} + \beta y - \beta y^2 = 0\cdot \tag{3.3}$$

By making the substitution $z(y) = \dfrac{d}{dy}v(y)$, Equation ((3.3)) can be reduced to the following first-order equation

$$\frac{-\frac{d}{dy}z(y)}{z(y)^3} + \beta y - \beta y^2 = 0\cdot$$

Consequently, the equilibrium solution is

$$\varphi_0(x) = \frac{3}{2}\sec^2\left(\frac{\sqrt{\beta}(x - A_2)}{2})\right),$$

where A_2 is an arbitrary constant. Figure 1 shows the equilibrium solutions for various values of A_2 and β.

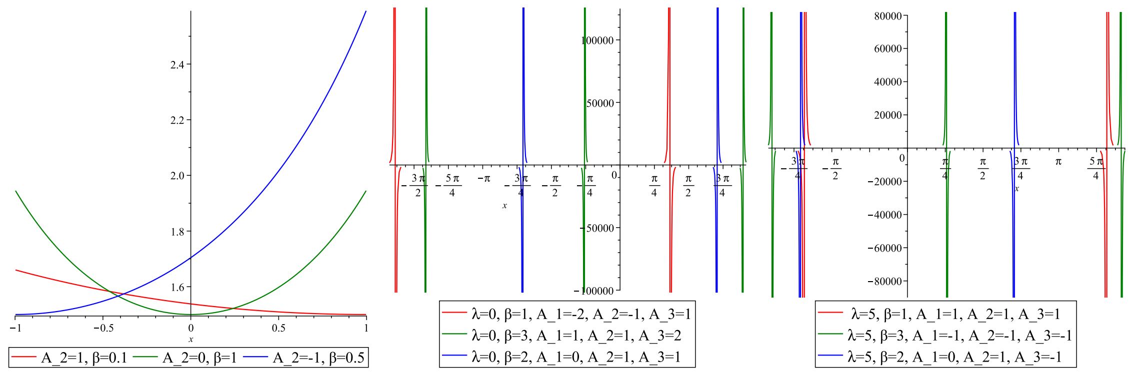

FIGURE 1. Left: Equilibrium solutions of Fisher equation. Center: Eigenfunction associated with $\lambda = 0$. Right: Eigenfunction associated with $\lambda = 5$.

The eigenvalue problem for the computed equilibrium solution $\varphi_0(x)$ takes the following form

$$\frac{d^2}{dx^2}\mu + \left(\beta - 3\beta\left(\sec^2\left(\frac{\sqrt{\beta}\ (x - A_2)}{2}\right)\right)\right)\mu = \lambda\mu\cdot \tag{3.4}$$

The solution (λ, μ) of Equation ((3.4)) represents a pair of eigenvalue and eigenvector. We assume that λ is a constant. The eigenvalue problem (3.4) is a second-order linear ODE, and its solution is given by

$$\frac{1}{\sin\left(\sqrt{\beta}(A_2 - x)\right)^3}\Bigg(A_3\left(\cos\left(\sqrt{\beta}(A_2 - x)\right) - 1\right)^2 \text{hypergeom}\Bigg(\left[\frac{-\sqrt{\beta} + \sqrt{\beta - \lambda}}{\sqrt{\beta}}, -\frac{\sqrt{\beta} + \sqrt{\beta - \lambda}}{\sqrt{\beta}}\right],$$
$$\left[-\frac{5}{2}\right], \frac{\cos\left(\sqrt{\beta}(A_2 - x)\right)}{2} + \frac{1}{2}\Bigg)\Bigg) + A_4\sqrt{2\cos\left(\sqrt{\beta}(A_2 - x)\right) - 2}\Bigg(\cos\left(\sqrt{\beta}(A_2 - x)\right) + 1)^2 \text{hypergeom}\Bigg($$
$$\left[\frac{-5\sqrt{\beta} + 2\sqrt{\beta - \lambda}}{2\sqrt{\beta}}, \frac{5\sqrt{\beta} + 2\sqrt{\beta - \lambda}}{2\sqrt{\beta}}\right], \left[-\frac{5}{2}\right], \frac{\cos\left(\sqrt{\beta}(A_2 - x)\right)}{2} + \frac{1}{2}\Bigg)\Bigg). \tag{3.5}$$

We suppose eigenvalues are real numbers. The hypergeometric function hypergeom$([r, s], [t], z)$ is a solution of the hypergeometric differential equation

$$z(1 - z)\frac{d^2w}{dz^2} + [t - (r + s + 1)z]\frac{dw}{dz} - rsw = 0\cdot$$

Theorem 3.1. *The number of eigenvalues of Equation ((3.4)) is infinite and the equilibrium solution $u^*(x)$ is a non-hyperbolic equilibrium solution.*

Proof. In the computed eigenfunction, $\sqrt{\beta - \lambda}$ exists, then $\lambda \leq \beta$. Therefore, the number of eigenvalues is infinite. Since zero is an eigenvalue, then the equilibrium solution $u^*(x)$ is a non-hyperbolic equilibrium solution. □

Figure 1 displays two eigenfunctions.

3.2. **The solutions of the invariance equation.** After calculating the equilibrium solution of the Fisher equation and the solutions of the eigenvalue problem, it is time to solve the invariance equation associated with Fisher equation. Let λ be a constant eigenvalue. Our goal is to find a function $\Phi(\theta, x)$ that satisfies the invariance equation

$$\Delta_F: \quad \lambda\theta \frac{\partial \Phi(\theta, x)}{\partial \theta} = \frac{\partial^2 \Phi(\theta, x)}{\partial x^2} + \beta\Phi(\theta, x)(1 - \Phi(\theta, x))\cdot \tag{3.6}$$

Let us consider a Lie group of infinitesimal transformations for the invariance equation ((3.6)) given by

$$\begin{aligned} \widehat{\theta} &= \theta + \sigma\kappa(\theta, x, \Phi) + O(\sigma^2), \\ \widehat{x} &= x + \sigma\zeta(\theta, x, \Phi) + O(\sigma^2), \\ \widehat{\Phi} &= \Phi + \sigma\eta(\theta, x, \Phi) + O(\sigma^2)\cdot \end{aligned}$$

The group parameter is σ and κ, ζ, and η are the infinitesimals for θ, x, and Φ respectively. The vector field associated with the given Lie symmetry group is given by

$$W = \kappa(\theta, x, \Phi)\frac{\partial}{\partial \theta} + \zeta(\theta, x, \Phi)\frac{\partial}{\partial x} + \eta(\theta, x, \Phi)\frac{\partial}{\partial \Phi}\cdot$$

The invariance condition is gained by applying $Pr^{(2)}W$ to (3.6). A system of PDEs is obtained by extracting the coefficients of the partial derivative of $\Phi(\theta, x)$ with respect to θ and x in the condition $Pr^{(2)}(\Delta_F)|_{\Delta_F=0} = 0$. The solution of this system is

$$\{\kappa(\theta, x, \Phi) = A_5\theta, \zeta(\theta, x, \Phi) = A_6, \eta(\theta, x, \Phi) = 0\}, \tag{3.7}$$

where A_5 and A_6 are arbitrary constants. Thus, the vector fields $\left\{W_1 = \theta\frac{\partial}{\partial\theta}, W_2 = \frac{\partial}{\partial x}\right\}$ generate a Lie algebra of infinitesimal symmetries. Therefore, the infinitesimal generators can be expressed as $\alpha_1\theta\frac{\partial}{\partial\theta} + \alpha_2\frac{\partial}{\partial x}$, where α_1 and α_2 are real numbers.

The one-parameter Lie groups $G_1, G_2 : (\theta, x, \Phi) \longrightarrow (\widehat{\theta}, \hat{x}, \widehat{\Phi})$, associated with the infinitesimal generators $W_1 = \theta\frac{\partial}{\partial\theta}$ and $W_2 = \frac{\partial}{\partial x}$, respectively, are defined as follows

$$G_1 : \exp(\sigma W_1)(\theta, x, \Phi) = (\widehat{\theta}, \hat{x}, \widehat{\Phi}), \quad G_2 : \exp(\sigma W_2)(\theta, x, \Phi) = (\widehat{\theta}, \hat{x}, \widehat{\Phi})\cdot$$

Therefore,

$$G_1 : \exp(\sigma W_1)(\theta, x, \Phi) = (e^{\sigma}\theta, x, \Phi), \quad G_2 : \exp(\sigma W_2)(\theta, x, \Phi) = (\theta, x + \sigma, \Phi)\cdot$$

Let $\Phi(\theta, x)$ be a solution to the invariance equation (3.6). Therefore the functions $\Phi(e^{-\sigma}\theta, x)$ and $\Phi(\theta, x - \sigma)$ are solutions of Eq. (3.6).

3.2.1. *One-dimensional optimal system.* The Lie bracket operation on the set of infinitesimal symmetries forms a Lie algebra, in which the Lie bracket of two symmetries, W_1 and W_2, is given by $[W_1, W_2] = W_1W_2 - W_2W_1$. Table 1 displays the commutator Table and Adjoint Table. To construct the Adjoint Table, we use the Adjoint action defined by

$$\mathrm{Ad}(\exp(\sigma W_i)W_j) = W_j - \sigma[W_i, W_j] + \frac{1}{2}\sigma^2[W_i, [W_i, W_j]] - \cdots$$

Table 1. Commutator Table and Adjoint Table.

$[W_i, W_j]$	W_1	W_2
W_1	0	0
W_2	0	0

$\mathrm{Ad}(\exp(\varepsilon W_i)W_j)$	W_1	W_2
W_1	W_1	W_2
W_2	W_1	W_2

Using Table 1, we can see that the vector fields $\{W_1, W_2\}$ form an abelian Lie algebra. Therefore, the one-dimensional optimal system is given by

$$W_1' = \langle W_1 = \theta\frac{\partial}{\partial\theta}\rangle, \quad W_2' = \langle W_1 + \alpha W_2 = \theta\frac{\partial}{\partial\theta} + \alpha\frac{\partial}{\partial x}\rangle, \tag{3.8}$$

where α is an arbitrary real number.

3.2.2. *Group invariant solution of invariance Equation* (3.6)*.* The characteristic equation for the vector field $W = \kappa(\theta, x, \Phi)\frac{\partial}{\partial\theta} + \zeta(\theta, x, \Phi)\frac{\partial}{\partial x} + \eta(\theta, x, \Phi)\frac{\partial}{\partial\Phi}$ is given by

$$\frac{d\theta}{\kappa(\theta, x, \Phi)} = \frac{dx}{\zeta(\theta, x, \Phi)} = \frac{d\Phi}{\eta(\theta, x, \Phi)}\cdot$$

The associated similarity variables and similarity solutions are acquired by utilizing characteristic equations of vector fields W_1' and W_2'.

3.3. **Symmetry reduction with $\theta\frac{\partial}{\partial\theta}$.** The characteristic equation for $\theta\frac{\partial}{\partial\theta}$ is $\frac{d\theta}{\theta} = \frac{dx}{0} = \frac{d\Phi}{0}$. Therefore, the similarity variable is $s = \log(\theta)$. After substituting the similarity variable into Equation (3.6), we can reduce it to the following equation with independent variables s and x, given by

$$\lambda\frac{\partial\Phi(s, x)}{\partial s} = \frac{\partial^2\Phi(s, x)}{\partial x^2} + \beta\Phi(s, x)(1 - \Phi(s, x))\cdot \tag{3.9}$$

We employ the tanh method to solve Equation (3.9). We look for traveling wave solutions as follows

$$\Phi(s, x) = \Phi(\vartheta), \quad \vartheta = c(x - vs), \tag{3.10}$$

such that $\frac{\partial}{\partial s} = -vc\frac{d}{d\vartheta}$ and $\frac{\partial^2}{\partial x^2} = c^2\frac{d^2}{d\vartheta^2}$. We introduce $Z = \tanh(\vartheta)$ as a new independent variable. Therefore,

$$\begin{aligned}\frac{d}{d\vartheta} &= (1 - Z^2)\frac{d}{dZ},\\ \frac{d^2}{d\vartheta^2} &= (1 - Z^2)^2\frac{d^2}{dZ^2} - 2Z(1 - Z^2)\frac{d}{dZ}\cdot\end{aligned}$$

Equation (3.9) is reduced to the following equation

$$c^2(1 - Z^2)^2\frac{d^2\Phi}{dZ^2} + (\lambda vc - 2Zc^2)(1 - Z^2)\frac{d\Phi}{dZ} + \beta\Phi(1 - \Phi) = 0\cdot \tag{3.11}$$

Suppose the solution of Equation (3.11) is the series expansion

$$\Phi(Z) = \sum_{i=0}^{l} a_i Z^i + \sum_{i=1}^{l} b_i Z^{-i},$$

where the coefficients a_i and b_i are unknown for $i = 1\cdots l$. By comparing the highest order of the available derivative and the highest available power of Φ in Equation (3.11), we conclude that $l = 2$. Consequently, the solution of Equation (3.11) is

$$\Phi(Z) = a_0 + a_1 Z + a_2 Z^2 + \frac{b_1}{Z} + \frac{b_2}{Z^2}\cdot \tag{3.12}$$

By substituting (3.12) in Equation (3.11), a system of algebraic equations is gained by extracting the coefficients of Z. The solutions to the system of algebraic equations were obtained using Maple software version 2022 and are displayed in Table 2.

TABLE 2. The solutions of the produced system by extracting coefficients Z.

Case	a_0	a_1	a_2	b_1	b_2	λ	β	c	v
2-1	$-1/2$	0	0	0	$3/2$	0	$4c^2$	arbitrary	arbitrary
2-2	$3/2$	0	0	0	$-3/2$	0	$-4c^2$	arbitrary	arbitrary
2-3	$1/4$	0	0	$1/2$	$1/4$	$-10c/v$	$24c^2$	arbitrary	arbitrary
2-4	$3/4$	0	0	$-1/2$	$-1/4$	$-10c/v$	$-24c^2$	arbitrary	arbitrary
2-5	$1/4$	0	0	$-1/2$	$1/4$	$10c/v$	$24c^2$	arbitrary	arbitrary
2-6	$3/4$	0	0	$1/2$	$-1/4$	$10c/v$	$-24c^2$	arbitrary	arbitrary
2-7	$-1/2$	0	0	0	$3/2$	arbitrary	$4c^2$	arbitrary	0
2-8	$3/2$	0	0	0	$-3/2$	arbitrary	$-4c^2$	arbitrary	0
2-9	0	0	0	0	0	arbitrary	arbitrary	arbitrary	arbitrary
2-10	arbitrary	0	0	0	0	arbitrary	0	arbitrary	arbitrary
2-11	1	0	0	0	0	arbitrary	arbitrary	arbitrary	arbitrary
2-12	$-1/2$	0	$3/2$	0	0	0	$4c^2$	arbitrary	arbitrary
2-13	$3/2$	0	$-3/2$	0	0	0	$-4c^2$	arbitrary	arbitrary
2-14	$1/4$	$1/2$	$1/4$	0	0	$-10c/v$	$24c^2$	arbitrary	arbitrary
2-15	$3/4$	$-1/2$	$-1/4$	0	0	$-10c/v$	$-24c^2$	arbitrary	arbitrary
2-16	$1/4$	$-1/2$	$1/4$	0	0	$10c/v$	$24c^2$	arbitrary	arbitrary
2-17	$3/4$	$1/2$	$-1/4$	0	0	$10c/v$	$-24c^2$	arbitrary	arbitrary
2-18	$-1/2$	0	$3/2$	0	0	arbitrary	$4c^2$	arbitrary	arbitrary
2-19	$3/2$	0	$-3/2$	0	0	arbitrary	$-4c^2$	arbitrary	0
2-20	arbitrary	arbitrary	arbitrary	arbitrary	arbitrary	arbitrary	0	0	arbitrary
2-21	$1/4$	0	$3/8$	0	$3/8$	0	$16c^2$	arbitrary	arbitrary
2-22	$3/4$	0	$-3/8$	0	$-3/8$	0	$-16c^2$	arbitrary	arbitrary
2-23	$3/8$	$1/4$	$1/16$	$1/4$	$1/16$	$-20c/v$	$96c^2$	arbitrary	arbitrary
2-24	$5/8$	$-1/4$	$-1/16$	$-1/4$	$-1/16$	$-20c/v$	$-96c^2$	arbitrary	arbitrary
2-25	$3/8$	$-1/4$	$1/16$	$-1/4$	$1/16$	$20c/v$	$96c^2$	arbitrary	arbitrary
2-26	$5/8$	$1/4$	$-1/16$	$1/4$	$-1/16$	$20c/v$	$-96c^2$	arbitrary	arbitrary
2-27	$1/4$	0	$3/8$	0	$3/8$	arbitrary	$16c^2$	arbitrary	0
2-28	$3/4$	0	$-3/8$	0	$-3/8$	arbitrary	$-16c^2$	arbitrary	0

In case 2-1, c and v are arbitrary. Since the constant $\beta > 0$, then this solution is acceptable. Since $\lambda = 0$, therefore the invariant manifold is a center manifold. Then $c = \pm\dfrac{\sqrt{\beta}}{2}$ and

$$\Phi(\theta, x) = -\frac{1}{2} + \frac{3}{2}\frac{1}{\tanh^2\left(\pm\dfrac{\sqrt{\beta}}{2}\,(x - v\log(\theta))\right)}.$$

Remark 3.2. In the cases 2-2, 2-4, 2-6, 2-8, 2-13, 2-15, 2-17, 2-19, 2-22, 2-24, 2-26, and 2-28, $\beta < 0$. For this reason, the obtained solutions are not acceptable. In cases 2-10 and 2-20, $\beta = 0$. From this, the obtained solutions are not acceptable.

In case 2-3, $c = \pm\dfrac{\sqrt{6\beta}}{12}$ and v is arbitrary, then $\lambda = -\dfrac{10c}{v}$ can be a positive number. Therefore, the computed manifolds can be unstable manifolds with

$$\Phi(\theta, x) = \frac{1}{4} + \frac{\frac{1}{2}}{\tanh(\pm\frac{\sqrt{6\beta}}{12}(x - v\log(\theta)))} + \frac{\frac{1}{4}}{\tanh^2(\pm\frac{\sqrt{6\beta}}{12}(x - v\log(\theta)))}.$$

Other cases are expressed in Table 3.

TABLE 3. The solutions of Equation (3.6).

Case	Eigenvalue	Type of invariant manifold	$U(\theta,x)$
3-1	$\lambda=0$	center	$-\frac{1}{2}+\frac{\frac{3}{2}}{\tanh^2\left(\pm\frac{\sqrt{\beta}}{2}(x-v\log(\theta))\right)}$
3-2	$\lambda=\frac{5\sqrt{6\beta}}{6v}$	unstable	$\frac{1}{4}+\frac{\frac{1}{2}}{\tanh(\pm\frac{\sqrt{6\beta}}{12}(x-v\log(\theta)))}+\frac{\frac{1}{4}}{\tanh^2(\pm\frac{\sqrt{6\beta}}{12}(x-v\log(\theta)))}$
3-3	$\lambda=\frac{5\sqrt{6\beta}}{6v}$	unstable	$\frac{1}{4}+\frac{-\frac{1}{2}}{\tanh(\pm\frac{5\sqrt{6\beta}}{6v}(x-v\log(\theta)))}+\frac{\frac{1}{4}}{\tanh^2(\pm\frac{5\sqrt{6\beta}}{6v}(x-v\log(\theta)))}$
3-4	arbitrary	stable and unstable and center	$-\frac{1}{2}+\frac{\frac{3}{2}}{\tanh^2(\pm\frac{\sqrt{\beta}}{2}x)}$
3-5	arbitrary	stable and unstable and center	0
3-6	arbitrary	stable and unstable and center	1
3-7	$\lambda=0$	center	$-\frac{1}{2}+\frac{3}{2}\tanh^2(\pm\frac{\sqrt{\beta}}{2}(x-v\log(\theta)))$
3-8	$\lambda=\frac{5\sqrt{6\beta}}{6v}$	unstable	$\frac{1}{4}+\frac{1}{2}\tanh(\pm\frac{5\sqrt{6\beta}}{6v}(x-v\log(\theta)))+\frac{1}{4}\tanh^2(\pm\frac{5\sqrt{6\beta}}{6v}(x-v\log(\theta)))$
3-9	$\lambda=\frac{5\sqrt{6\beta}}{6v}$	unstable	$\frac{1}{4}-\frac{1}{2}\tanh(\pm\frac{5\sqrt{6\beta}}{6v}(x-v\log(\theta)))+\frac{1}{4}\tanh^2(\pm\frac{5\sqrt{6\beta}}{6v}(x-v\log(\theta)))$
3-10	arbitrary	stable and unstable and center	$-\frac{1}{2}+\frac{3}{2}\tanh^2(\pm\frac{\sqrt{\beta}}{2}x)$
3-11	$\lambda=0$	center	$\frac{1}{4}+\frac{3}{8}\tanh^2(\pm\frac{\sqrt{\beta}}{4}(x-v\log(\theta)))+\frac{\frac{3}{8}}{\tanh^2(\pm\frac{5\sqrt{\beta}}{\sqrt{6v}}(x-v\log(\theta)))}$
3-12	$\lambda=\frac{5\sqrt{\beta}}{\sqrt{6v}}$	unstable	$\frac{3}{8}+\frac{1}{4}\tanh(\pm\frac{5\sqrt{\beta}}{\sqrt{6v}}(x-v\log(\theta)))+\frac{1}{16}\tanh^2(\pm\frac{5\sqrt{\beta}}{\sqrt{6v}}(x-v\log(\theta)))+\frac{\frac{1}{4}}{\tanh(\pm\frac{5\sqrt{\beta}}{\sqrt{6v}}(x-v\log(\theta)))}+\frac{\frac{1}{16}}{\tanh^2(\pm\frac{5\sqrt{\beta}}{\sqrt{6v}}(x-v\log(\theta)))}$
3-13	$\lambda=\frac{5\sqrt{\beta}}{\sqrt{6v}}$	unstable	$\frac{3}{8}-\frac{1}{4}\tanh(\pm\frac{5\sqrt{\beta}}{\sqrt{6v}}(x-v\log(\theta)))+\frac{1}{16}\tanh^2(\pm\frac{5\sqrt{\beta}}{\sqrt{6v}}(x-v\log(\theta)))+\frac{-\frac{1}{4}}{\tanh(\pm\frac{5\sqrt{\beta}}{\sqrt{6v}}(x-v\log(\theta)))}+\frac{\frac{1}{16}}{\tanh^2(\pm\frac{5\sqrt{\beta}}{\sqrt{6v}}(x-v\log(\theta)))}$
3-14	arbitrary	stable and unstable and center	$\frac{1}{4}+\frac{3}{8}\tanh^2(\pm\frac{\sqrt{\beta}}{4}x))+\frac{\frac{3}{8}}{\tanh^2(\pm\frac{\sqrt{\beta}}{4}x))}$

Based on the analysis of cases 2-1, 2-12, and 2-21, it can be concluded that the parameterized manifolds in these cases are center manifolds. According to case 2-7, the invariant manifold is only function $\Phi(x)=-\frac{1}{2}+\frac{\frac{3}{2}}{\tanh^2(\pm\frac{\sqrt{\beta}}{2}x)}$. Since λ is arbitrary, this function can be a stable manifold, an unstable manifold, or a center manifold. Also, the parameterized manifolds corresponding to cases 2-9, 2-11, 2-18 and 2-27 are the single functions such that are displayed in Table 3. These manifolds can be of any type.

Theorem 3.3. *Based on the analysis of cases 2-3, 2-5, 2-14, 2-16, 2-23, and 2-25, it can be concluded that the parameterized manifolds in these cases are unstable manifolds.*

Proof. 1) In case 2-3, we have $c = \pm\frac{\sqrt{6\beta}}{12}$ and $\lambda = -\frac{10c}{v}$.

(a) Let $c = \frac{\sqrt{6\beta}}{12}$, then $\lambda = -\frac{10(\frac{\sqrt{6\beta}}{12})}{v} \leq \beta$. We gain $v \leq \frac{-\frac{5}{6}\sqrt{6\beta}}{\beta}$ and v is negative. Therefore, λ is positive.

(b) Let $c = -\frac{\sqrt{6\beta}}{12}$, then $\lambda = -\frac{10(-\frac{\sqrt{6\beta}}{12})}{v} \leq \beta$. The result is that $v \geq \frac{\frac{5}{6}\sqrt{6\beta}}{\beta}$, and v is positive. Therefore, λ is positive.

2) In case 2-5, $c = \pm\frac{\sqrt{6\beta}}{12}$ and $\lambda = \frac{10c}{v}$.

(a) Suppose $c = \frac{\sqrt{6\beta}}{12}$, then $\lambda = \frac{10(\frac{\sqrt{6\beta}}{12})}{v} \leq \beta$. We deduce that $v \geq \frac{\frac{5}{6}\sqrt{6\beta}}{\beta}$, and v is positive. Therefore, λ is positive.

(b) Suppose $c = -\frac{\sqrt{6\beta}}{12}$, then $\lambda = \frac{10(-\frac{\sqrt{6\beta}}{12})}{v} \leq \beta$. We deduce $v \leq \frac{-\frac{5}{6}\sqrt{6\beta}}{\beta}$, and v is negative. Therefore, λ is positive.

3) Cases 2-14 and 2-16 are similar to cases 2-3 and 2-5, respectively.

4) In case 2-23, $c = \pm\frac{\sqrt{6\beta}}{24}$ and $\lambda = -\frac{20c}{v}$.

(a) Let $c = \frac{\sqrt{6\beta}}{24}$, then we obtain that $\lambda = -\frac{20(\frac{\sqrt{6\beta}}{24})}{v}$. Consequently, $v \leq \frac{-\frac{5}{6}\sqrt{6\beta}}{\beta}$, and v is negative. Therefore, $\lambda > 0$.

(b) Let $c = -\frac{\sqrt{6\beta}}{24}$, then $\lambda = -\frac{20(-\frac{\sqrt{6\beta}}{24})}{v}$. Consequently, $v \geq \frac{\frac{5}{6}\sqrt{6\beta}}{\beta}$, and v is positive. Therefore, $\lambda > 0$.

5) In case 2-25, $c = \pm\frac{\sqrt{6\beta}}{24}$ and $\lambda = \frac{20c}{v}$.

(a) Let $c = \frac{\sqrt{6\beta}}{24}$, then $\lambda = \frac{20(\frac{\sqrt{6\beta}}{24})}{v}$. Therefore, $v \geq \frac{\frac{5}{6}\sqrt{6\beta}}{\beta}$, and $v > 0$. Also, $\lambda > 0$.

(b) Let $c = -\frac{\sqrt{6\beta}}{24}$, then $\lambda = \frac{20(-\frac{\sqrt{6\beta}}{24})}{v}$. Therefore, $v \leq \frac{-\frac{5}{6}\sqrt{6\beta}}{\beta}$, and $v > 0$. Consequently, $\lambda > 0$.

□

3.4. Symmetry reduction with $\theta\frac{\partial}{\partial\theta} + \alpha\frac{\partial}{\partial x}$.

The characteristic equations associated with vector field $\theta\frac{\partial}{\partial\theta} + \alpha\frac{\partial}{\partial x}$ are $\frac{d\theta}{\theta} = \frac{dx}{\alpha}$. Then, the similarity variable is $y = \theta^{\alpha}\exp(-x)$. The invariance equation (3.6) reduces to

$$y^2\frac{d^2\Phi}{dy^2} + (1-\lambda\alpha)y\frac{d\Phi}{dy} + \beta\Phi(1-\Phi) = 0\cdot \tag{3.13}$$

Vector field $y\dfrac{\partial}{\partial y}$ is an infinitesimal generator for Equation (3.13), and $z = \log(y)$ is the similarity variable. From this, we deduce that Equation (3.13) reduces to

$$\frac{d^2\Phi}{dz^2} + (2 - \lambda\alpha)\frac{d\Phi}{dz} + \beta\Phi(1 - \Phi) = 0\cdot \tag{3.14}$$

Let $V = \tanh(z)$. Then Equation (3.14) transform to

$$(1 - V^2)^2\frac{d^2\Phi}{dV^2} + (2 - \lambda\alpha - 2V)(1 - V^2)\frac{d\Phi}{dV} + \beta\Phi(1 - \Phi)\cdot \tag{3.15}$$

Suppose the solution of Equation (3.14) is the series expansion

$$\Phi(V) = \sum_{i=0}^{K} a_i V^i + \sum_{i=1}^{K} b_i V^{-i}\cdot$$

The coefficients a_i and b_i are unknown. By comparing the highest order of the available derivative and the highest available power of Φ in Equation (3.15), we conclude that $K = 2$. The solutions of Equation (3.15) are of the form

$$\Phi(V) = a_0 + a_1 V + a_2 V^2 + \frac{b_1}{V} + \frac{b_2}{V^2}\cdot \tag{3.16}$$

Substituting (3.16) in Equation (3.15) and extracting the coefficients of V, a system of algebraic equations is obtained. The unknowns are a_0, a_1, a_2, b_1, b_2, λ, and β. Solutions of Equation (3.6) are obtained from solving the system. The solutions to the discussed system are displayed in Table 4.

Table 4. The solutions of the produced system by extracting coefficients V.

Case	a_0	a_1	a_2	b_1	b_2	λ	β
$4-1$	0	0	0	0	0	arbitrary	arbitrary
$4-2$	arbitrary	0	0	0	0	arbitrary	0
$4-3$	1	0	0	0	0	arbitrary	arbitrary
$4-4$	$-1/2$	0	0	0	$3/2$	$2/\alpha$	4
$4-5$	$3/2$	0	0	0	$-3/2$	$2/\alpha$	-4
$4-6$	$-1/2$	0	$3/2$	0	0	$2/\alpha$	4
$4-7$	$3/2$	0	$-3/2$	0	0	$2/\alpha$	-4
$4-8$	$1/4$	0	$3/8$	0	$3/8$	$2/\alpha$	16
$4-9$	$3/4$	0	$-3/8$	0	$-3/8$	$2/\alpha$	-16
$4-10$	$1/4$	0	0	$-1/2$	$-1/4$	$-8/\alpha$	24
$4-11$	$1/4$	0	0	$1/2$	$1/4$	$12/\alpha$	24
$4-12$	$3/4$	0	0	$-1/2$	$-1/4$	$12/\alpha$	-24
$4-13$	$3/4$	0	0	$1/2$	$-1/4$	$-8/\alpha$	-24
$4-14$	$1/4$	$-1/2$	$1/4$	0	0	$-8/\alpha$	24
$4-15$	$3/4$	$1/2$	$-1/4$	0	0	$-8/\alpha$	-24
$4-16$	$1/4$	$1/2$	$1/4$	0	0	$12/\alpha$	24
$4-17$	$3/4$	$-1/2$	$-1/4$	0	0	$12/\alpha$	-24
$4-18$	$3/8$	$1/4$	$1/16$	$1/4$	$1/16$	$22/\alpha$	96
$4-19$	$5/8$	$-1/4$	$-1/16$	$-1/4$	$1/16$	$22/\alpha$	-96
$4-21$	$3/8$	$-1/4$	$1/16$	$-1/4$	$1/16$	$-18/\alpha$	96
$4-21$	$5/8$	$1/4$	$-1/16$	$1/4$	$-1/16$	$-18/\alpha$	-96

Remark 3.4. In case 4-2, $\beta = 0$ and in cases 4-5, 4-7, 4-9, 4-12, 4-13, 4-15, 4-17, 4-19, 4-21, $\beta < 0$. Therefore, in these cases, solutions are not acceptable. In cases 4-1 and 4-2, invariant manifolds contain only constant functions 0

and 1, respectively. In cases 4-4, 4-6, 4-8, 4-10, 4-11, 4-14, 4-16, 4-18, and 4-20, the eigenvalues are dependent on α, and as a result, λ can be both negative and positive. Therefore, in these cases, invariant manifolds can be stable or unstable.

The solutions associated with subalgebra $\langle\theta\frac{\partial}{\partial\theta}+\alpha\frac{\partial}{\partial x}\rangle$ are presented in Table 5.
The unstable manifolds introduced in case 3-2 are displayed in Figure 2. The center manifolds in case 3-1 are displayed in Figures 2 and 3. The unstable manifolds introduced in case 3-13 in Table 3 are displayed in Figure 3.

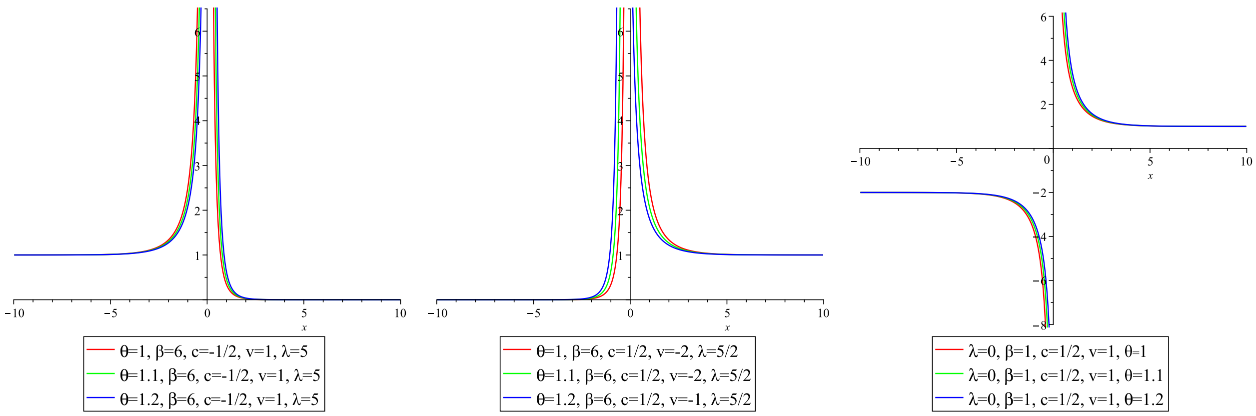

FIGURE 2. Left: Three functions on the unstable manifold in case 3-2 in Table 3 with $\lambda = 5$. Center: Three functions on the unstable manifold in case 3-2 in Table 3 with $\lambda = \frac{5}{2}$. Right: Three functions on the center manifold in case 3-1 in Table 3 with $\lambda = 0$.

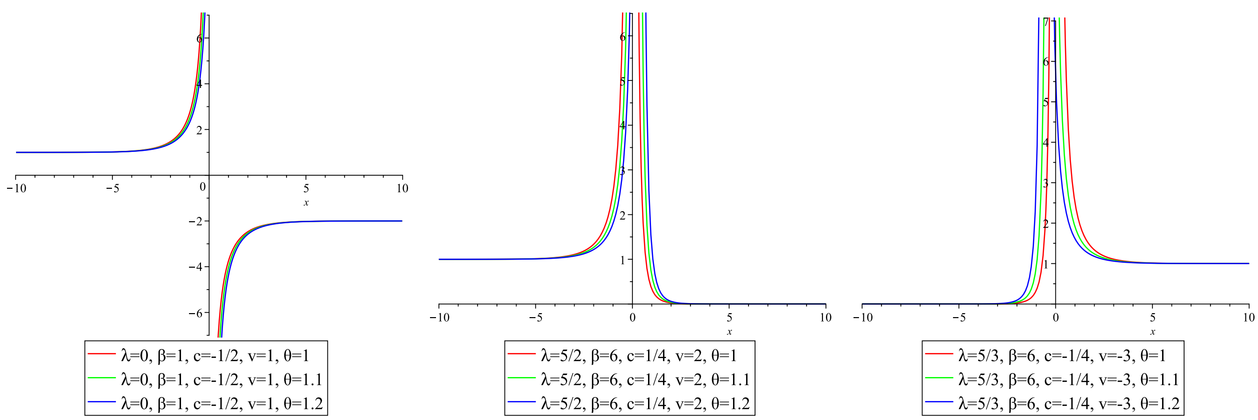

FIGURE 3. Left: Three functions on the center manifold of case 3-1 in Table 3 with $\lambda = 0$. Center:Three functions on the unstable manifold of case 13-3 in Table 3 with $\lambda = \frac{5}{2}$. Right: Three functions on the unstable manifold of case 13-3 in Table 3 with $\lambda = \frac{5}{3}$.

TABLE 5. The solutions of Equation (3.6).

Case	Eigenvalue	Type of invariant manifold	$U(\theta, x)$
5-1	arbitrary	stable or unstable or center	0
5-2	arbitrary	stable or unstable or center	1
5-3	$\frac{2}{\alpha}$	stable or unstable	$-\frac{1}{2}+\frac{\frac{3}{2}}{\tanh^2(\log(\theta^\alpha)-x)}$
5-4	$\frac{2}{\alpha}$	stable or unstable	$-\frac{1}{2}+\frac{3}{2}\tanh^2(\log(\theta^\alpha)-x)$
5-5	$\frac{2}{\alpha}$	stable or unstable	$\frac{1}{4}+\frac{3}{8}\tanh^2(\log(\theta^\alpha)-x)+\frac{\frac{3}{8}}{\tanh^2(\log(\theta^\alpha)-x)}$
5-6	$-\frac{8}{\alpha}$	stable or unstable	$\frac{1}{4}+\frac{-\frac{1}{2}}{\tanh(\log(\theta^\alpha)-x)}+\frac{-\frac{1}{4}}{\tanh^2(\log(\theta^\alpha)-x)}$
5-7	$\frac{12}{\alpha}$	stable or unstable	$\frac{1}{4}+\frac{\frac{1}{2}}{\tanh(\log(\theta^\alpha)-x)}+\frac{\frac{1}{4}}{\tanh^2(\log(\theta^\alpha)-x)}$
5-8	$-\frac{8}{\alpha}$	stable or unstable	$\frac{1}{4}-\frac{1}{2}\tanh(\log(\theta^\alpha)-x)+\frac{1}{4}\tanh^2(\log(\theta^\alpha)-x)$
5-9	$\frac{12}{\alpha}$	stable or unstable	$\frac{1}{4}+\frac{1}{2}\tanh(\log(\theta^\alpha)-x)+\frac{1}{4}\tanh^2(\log(\theta^\alpha)-x)$
5-10	$\frac{22}{\alpha}$	stable or unstable	$\frac{3}{8}+\frac{1}{4}\tanh(\log(\theta^\alpha)-x)+\frac{1}{16}\tanh^2(\log(\theta^\alpha)-x)+\frac{\frac{1}{4}}{\tanh(\log(\theta^\alpha)-x)}+\frac{\frac{1}{16}}{\tanh^2(\log(\theta^\alpha)-x)}$
5-11	$-\frac{18}{\alpha}$	stable or unstable	$\frac{3}{8}-\frac{1}{4}\tanh(\log(\theta^\alpha)-x)+\frac{1}{16}\tanh^2(\log(\theta^\alpha)-x)+\frac{-\frac{1}{4}}{\tanh(\log(\theta^\alpha)-x)}+\frac{\frac{1}{16}}{\tanh^2(\log(\theta^\alpha)-x)}$

The functions associated with the stable manifolds of case 5-6 in Table 5 and case 5-11 in Table 5 are displayed in Figure 4.

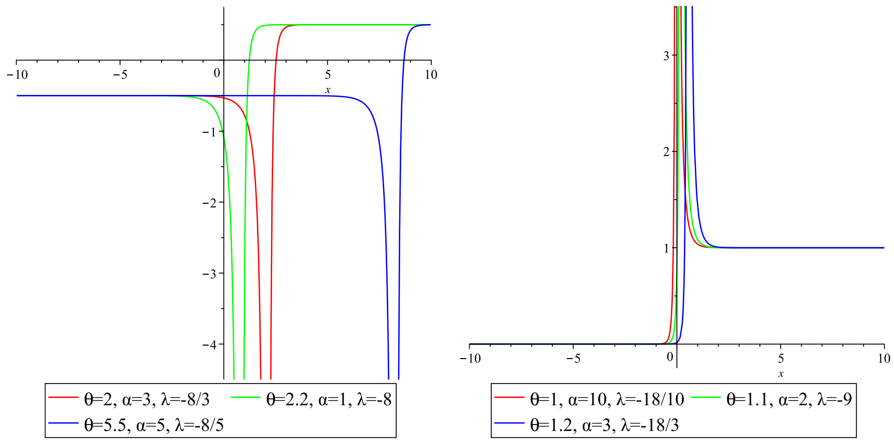

FIGURE 4. Left: Three functions on the stable manifold of case 5-6 in Table 5. Right: Three functions on the stable manifold of case 5-11 in Table 5.

4. Parametrizing invariant manifolds of Brain Tumor growth differential equation

One of the evolution equations studied in this paper is Brain tumor growth differential equation (glioblastomas) under medical treatment. Since we employ the parameterization method, we should solve an invariance equation. The solutions of the invariance equation will parameterize the invariant manifolds.

Brain Tumor growth differential equation is as follows

$$\frac{\partial Q}{\partial t} = \frac{1}{2}\frac{\partial^2 Q}{\partial x^2} + \exp(-Q) + \frac{1}{2}\exp(-2Q)\cdot \tag{4.1}$$

4.1. **Computing equilibrium solutions, eigenvalues and eigenfunctions.** First, we obtain the equilibrium solution of the evolution Equation (4.1). For this purpose, we need to solve the following ordinary differential equation

$$\frac{1}{2}\frac{\partial^2 Q}{\partial x^2} + \exp(-Q) + \frac{1}{2}\exp(-2Q) = 0\cdot \tag{4.2}$$

Let $\hat{x} = x + \sigma\xi(x, Q) + O(\sigma^2)$ and $\widehat{Q} = Q + \sigma\mu(x, Q) + O(\sigma^2)$ be the Lie group of transformations for Equation (4.2). By utilizing maple software, we acquire $\xi = A_1'$ and $\mu = 0$. The associated Lie algebra is $\langle \frac{\partial}{\partial x} \rangle$. By a similar method to solving Equation (3.1), we obtain the equilibrium solution of Equation (4.1). The equilibrium solutions are

$$Q_1^*(x) = -\frac{\log(2)}{2} + \log\left(\frac{\sqrt{A_2' + 2}\,\sin\left(\sqrt{A_2'}\,(A_3' - x)\sqrt{2}\right) + \sqrt{2}}{A_2'}\right), \tag{4.3}$$

$$Q_2^*(x) = -\frac{\log(2)}{2} + \log\left(\frac{-\sqrt{A_4' + 2}\,\sin\left(\sqrt{A_4'}\,(A_5' - x)\sqrt{2}\right) + \sqrt{2}}{A_4'}\right), \tag{4.4}$$

where A_2', A_3', A_4', and A_5' are arbitrary. The equilibrium solutions of Equation (4.1) are shown in Figure 5.

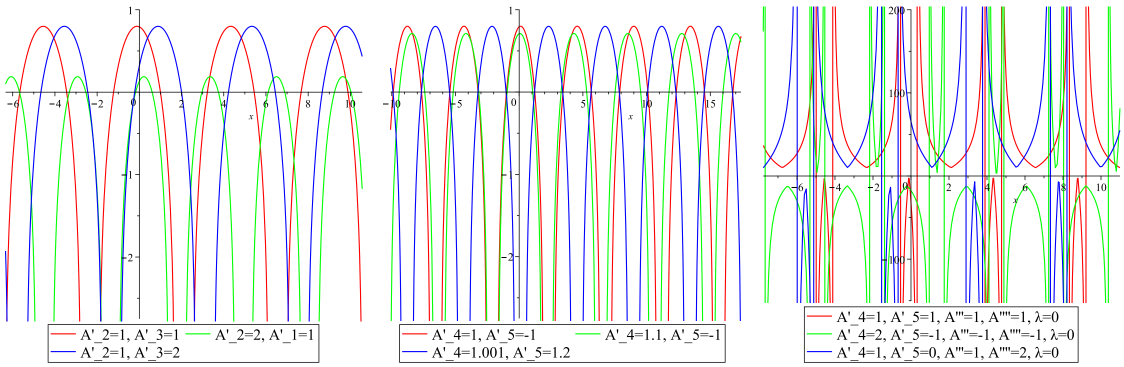

FIGURE 5. Left: Equilibrium solution Q_1^*. Center: Equilibrium solution Q_2^*. Right: Eigenfunction ϕ_2 associated with $\lambda = 0$

If we choose Q_1^* as the equilibrium solution, the eigenvalue problem is

$$\frac{1}{2}\frac{\partial^2 \phi_1}{\partial x^2} - \left(\frac{A_2'\sqrt{2}}{\sqrt{A_2'+2}\,\sin\left(\sqrt{A_2'}\,(A_3'-x)\,\sqrt{2}\right)+\sqrt{2}} + \frac{2{A_2'}^2}{\left(\sqrt{A_2'+2}\,\sin\left(\sqrt{A_2'}\,(A_3'-x)\,\sqrt{2}\right)+\sqrt{2}\right)^2} \right)\phi_1 = \lambda\phi_1. \tag{4.5}$$

Here λ is an eigenvalue and $\phi_1(x)$ is an eigenfunction. The linear differential Equation (4.5) is solved by Maple software, and the result is

$$\begin{aligned}
\phi_1(x) = {} & \left(2 + (A_2'+2)\left(\sin^2\left(\sqrt{A_2'}\,(A_3'-x)\sqrt{2}\right)\right) + 2\sqrt{2A_2'+4}\,\sin\left(\sqrt{A_2'}\,(A_3'-x)\sqrt{2}\right)\right)^{\frac{{A_2'}^2}{(\sqrt{2}-\sqrt{A_2'+2})^2(\sqrt{2}+\sqrt{A_2'+2})^2}} \\
& \Bigg(\sqrt{2\sin\left(\sqrt{A_2'}\,(A_3'-x)\sqrt{2}\right)+2}\,\mathrm{HeunG}\Bigg(-\frac{\sqrt{2}-\sqrt{A_2'+2}}{2\sqrt{A_2'+2}}, \frac{(36\sqrt{A_2'+2}\,\sqrt{2}+33A_2'+4\lambda+72){A_2'}^2}{8(\sqrt{2}+\sqrt{A_2'+2})^3(\sqrt{2}-\sqrt{A_2'+2})^2\sqrt{A_2'+2}}, \frac{5A_2'+2\sqrt{-\lambda A_2'}}{2A_2'}, \\
& \frac{(15A_2'+4\lambda+4\sqrt{-\lambda A_2'}){A_2'}^2}{6(\sqrt{2}+\sqrt{A_2'+2})^2(\sqrt{2}-\sqrt{A_2'+2})^2\left(A_2'+\frac{2\sqrt{-\lambda A_2'}}{3}\right)}, \frac{3}{2}, \frac{1}{2}, \frac{\sin(\sqrt{A_2'}\,(A_3'-x)\sqrt{2})}{2}+\frac{1}{2}\Bigg)A'' + \mathrm{HeunG}\Bigg(-\frac{\sqrt{2}-\sqrt{A_2'+2}}{2\sqrt{A_2'+2}}, \\
& \frac{((4A_2'+\lambda)\sqrt{A_2'+2}+(A_2'-\lambda)\sqrt{2}A_2'}{2\sqrt{A_2'+2}\,(\sqrt{2}-\sqrt{A_2'+2})^2(\sqrt{2}+\sqrt{A_2'+2})^2}, \frac{2A_2'+\sqrt{-\lambda A_2'}}{A_2'}, \frac{(6A_2'+2\lambda+\sqrt{-\lambda A_2'}){A_2'}^2}{(3A_2'+2\sqrt{-\lambda A_2'})(\sqrt{2}+\sqrt{A_2'+2})^2(\sqrt{2}-\sqrt{A_2'+2})^2}, \frac{1}{2}, \frac{1}{2}, \\
& \frac{\sin(\sqrt{A_2'}\,(A_3'-x)\sqrt{2})}{2}+\frac{1}{2}\Bigg)A'\Bigg).
\end{aligned} \tag{4.6}$$

Here A' and A'' are arbitrary constants. Suppose all eigenvalues are real.

The eigenvalue problem associated with equilibrium solution Q_2^* is given by

$$\frac{1}{2}\frac{\partial^2 \phi_2}{\partial x^2} - \left(\frac{\sqrt{2}\,A_4'}{-\sqrt{A_4'+2}\,\sin\left(\sqrt{A_4'}\,(A_5'-x)\,\sqrt{2}\right)+\sqrt{2}} + \frac{2{A_4'}^2}{\left(-\sqrt{A_4'+2}\,\sin\left(\sqrt{A_4'}\,(A_5'-x)\,\sqrt{2}\right)+\sqrt{2}\right)^2} \right)\phi_2 = \lambda\phi_2. \tag{4.7}$$

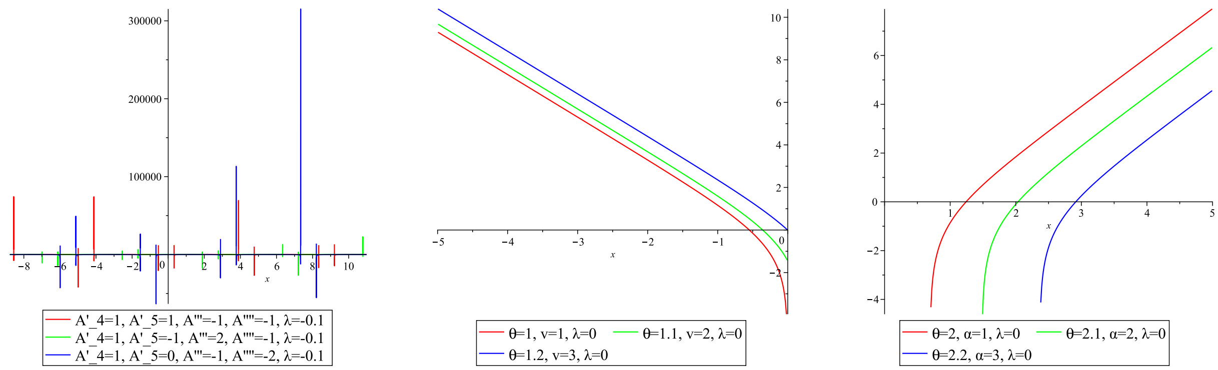

FIGURE 6. Left: Eigenfunction ϕ_2 associated with $\lambda = -0.1$. Center: Some functions of the center manifold of case 7-4 with $\lambda = 0$ in Table 7. Right: Some functions of the center manifold of case 9-2 with $\lambda = 0$ in Table 8 on the right side.

The solution (λ, ϕ_2) of Equation (4.7) represents a pair of eigenvalue and eigenfunction. Maple software was used to solve the linear differential Equation (4.7), and the resulting solution is

$$\begin{aligned}\phi_2(x) = {}& \left(-\sqrt{A_4'+2}\,\sin\left(\sqrt{A_4'}\,(A_5'-x)\sqrt{2}\right)+\sqrt{2}\right)^{\frac{2{A_4'}^2}{\left(\sqrt{2}+\sqrt{A_4'+2}\right)^2\left(\sqrt{2}-\sqrt{A_4'+2}\right)^2}} \left(\sqrt{2\sin\left(\sqrt{A_4'}\,(A_5'-x)\sqrt{2}\right)+2}\,\mathrm{HeunG}\left(\frac{\sqrt{2}+\sqrt{A_4'+2}}{2\sqrt{A_4'+2}},\right.\right.\\
& -\frac{\left(-36\sqrt{2}\sqrt{A_4'+2}+33A_4'+4\lambda+72\right){A_4'}^2}{8\left(\sqrt{2}+\sqrt{A_4'+2}\right)^2\left(\sqrt{2}-\sqrt{A_4'+2}\right)^3\sqrt{A_4'+2}}, \frac{5A_4'+2\sqrt{-\lambda A_4'}}{2A_4'}, \frac{\left(15A_4'+4\lambda+4\sqrt{-\lambda A_4'}\right){A_4'}^2}{6\left(A_4'+\frac{2\sqrt{-\lambda A_4'}}{3}\right)\left(\sqrt{2}+\sqrt{A_4'+2}\right)^2\left(\sqrt{2}-\sqrt{A_4'+2}\right)^2}, \frac{3}{2}, \frac{1}{2},\\
& \left.\frac{\sin\left(\sqrt{A_4'}\,(A_5'-x)\sqrt{2}\right)}{2}+\frac{1}{2}\right)A'''' + \mathrm{HeunG}\left(\frac{\sqrt{2}+\sqrt{A_4'+2}}{2\sqrt{A_4'+2}}, -\frac{A_4'\left(\left(-4A_4'-\lambda\right)\sqrt{A_4'+2}+\sqrt{2}\left(A_4'-\lambda\right)\right)}{2\sqrt{A_4'+2}\left(\sqrt{2}+\sqrt{A_4'+2}\right)^2\left(\sqrt{2}-\sqrt{A_4'+2}\right)^2}, \frac{2A_4'+\sqrt{-\lambda A_4'}}{A_4'},\right.\\
& \left.\left.\frac{\left(6A_4'+2\lambda+\sqrt{-\lambda A_4'}\right){A_4'}^2}{\left(3A_4'+2\sqrt{-\lambda A_4'}\right)\left(\sqrt{2}+\sqrt{A_4'+2}\right)^2\left(\sqrt{2}-\sqrt{A_4'+2}\right)^2}, \frac{1}{2}, \frac{1}{2}, \frac{\sin\left(\sqrt{A_4'}\,(A_5'-x)\sqrt{2}\right)}{2}+\frac{1}{2}\right)A'''\right)\cdot\end{aligned} \tag{4.8}$$

Where A''' and A'''' are arbitrary constants. Figure 5 displays the eigenfunction ϕ_1 for various values of A_2', A_3', A', A'', and λ. The eigenfunction ϕ_2 for various values of A_4', A_5', A''', A'''' and λ is displayed in Figure 6. The function $\mathrm{HeuneG}(a, q, \omega, \psi, \gamma, \kappa, x)$ satisfies the general Heun differential equation given by

$$x(x-1)(x-a)\frac{d^2y}{dx^2} + ((x-1)(x-a)\gamma + x(x-a)\kappa + x(x-1)(1+\omega+\psi-\gamma-\kappa))\frac{dy}{dx} + (\omega\psi x - q)y = 0\cdot$$

4.2. **The solutions of the invariance equation.** Now we should solve the following invariance equation

$$\Delta_\Psi: \quad \lambda\theta\frac{\partial\Psi}{\partial\theta} = \frac{1}{2}\frac{\partial^2\Psi}{\partial x^2} + \exp(-\Psi) + \frac{1}{2}\exp(-2\Psi)\cdot \tag{4.9}$$

We consider the following Lie group of infinitesimal transformations with parameter σ

$$\begin{aligned}\widehat{\theta} &= \theta + \sigma\kappa(\theta, x, \Psi) + O(\sigma^2),\\ \widehat{x} &= x + \sigma\zeta(\theta, x, \Psi) + O(\sigma^2),\\ \widehat{\Psi} &= \Psi + \sigma\eta(\theta, x, \Psi) + O(\sigma^2),\end{aligned}$$

which act on the variables θ, x, and Ψ. The elements of the Lie algebra of infinitesimal symmetries have the form

$$W = \kappa(\theta, x, \Psi)\frac{\partial}{\partial\theta} + \zeta(\theta, x, \Psi)\frac{\partial}{\partial x} + \eta(\theta, x, \Psi)\frac{\partial}{\partial\Psi}\cdot$$

The invariance condition is

$$Pr^{(2)}W(\Delta_\Psi)|_{\Delta_\Psi=0} = 0, \tag{4.10}$$

which $Pr^{(2)}W$ represents the second-order prolongation of W. We obtain a system of PDEs by extracting the coefficients of the partial derivatives of $\Psi(\theta, x)$ with respect to θ and x. The solution of this system is given by

$$\kappa(\theta, x, \Psi) = D_1\theta, \quad \zeta(\theta, x, \Psi) = D_2, \quad \eta(\theta, x, \Psi) = 0,$$

where $D_1, D_2 \in \mathbb{R}$ are arbitrary. Consequently, the infinitesimal symmetries of the invariance Equation (4.9) are $\alpha_1\theta\dfrac{\partial}{\partial\theta} + \alpha_2\dfrac{\partial}{\partial x}$. Where $\alpha_1, \alpha_2 \in \mathbb{R}$ are arbitrary.

Remark 4.1. The one-parameter Lie groups of Equation (4.9) are the same as the one-parameter Lie groups of Equation (3.6). Let $\Psi(\theta, x)$ is a solution of the invariance equation (4.9). Therefore, the functions $\Psi(e^{-\sigma}\theta, x)$ and $\Psi(\theta, x - \sigma)$ are solutions of Equation (4.9).

Remark 4.2. Similar to Section 3.2.1, the obtained Lie algebra is abelian. Therefore the one-dimensional optimal system is

$$W_1'' := \langle\theta\frac{\partial}{\partial\theta}\rangle, \quad W_2'' := \langle\theta\frac{\partial}{\partial\theta} + \alpha\frac{\partial}{\partial x}\rangle\cdot$$

Symmetry reduction with $\theta\dfrac{\partial}{\partial\theta}$. As described in subsection 3.2.2, for the vector field $\theta\dfrac{\partial}{\partial\theta}$, the similarity variable is given by $s = \log(\theta)$. We substitute the variable $s = \log(\theta)$ into Equation (4.9), resulting in

$$\lambda\frac{\partial\Psi}{\partial s} = \frac{1}{2}\frac{\partial^2\Psi}{\partial x^2} + \exp(-\Psi) + \frac{1}{2}\exp(-2\Psi)\cdot \tag{4.11}$$

By means of tanh method, we solve Equation (4.11). Considering the change of variable $\vartheta = c(x - vs)$, we can obtain

$$-\lambda cv\frac{\partial\Psi}{\partial\vartheta} = \frac{1}{2}c^2\frac{\partial^2\Psi}{\partial x^2} + \exp(-\Psi) + \frac{1}{2}\exp(-2\Psi)\cdot \tag{4.12}$$

Let $Z = \tanh(\vartheta)$, then

$$\frac{1}{2}c^2(1-Z^2)^2\frac{d^2\Psi}{dZ^2} + (\lambda cv - c^2Z)(1-Z^2)\frac{d\Psi}{dZ} + \exp(-\Psi) + \frac{1}{2}\exp(-2\Psi) = 0\cdot \tag{4.13}$$

Suppose the solution of Equation (4.13) is of the form $\Psi(Z) = \log\left(\dfrac{1}{S(Z)}\right)$, where $S(Z) = \sum_{i=0}^K a_iZ^i + \sum_{i=1}^K b_iZ^{-i}$. Equation (4.13) is reduced to

$$\frac{1}{2}c^2(1-Z^2)^2\left(\frac{-S''S + S'^2}{S^2}\right) + (\lambda cv - c^2Z)(1-Z^2)\left(\frac{-S'}{S}\right) + \exp(-S) + \frac{1}{2}\exp(-2S) = 0\cdot \tag{4.14}$$

To find the value of K, we consider the highest order of the available derivative and the highest available power of Ψ in Equation (4.14) and balance them. This yields $K = 1$. Therefore, the solution of Equation (4.14) is

$$S(Z) = -\log\left(a_0 + a_1Z + \frac{b_1}{Z}\right)\cdot \tag{4.15}$$

We substitute Equation (4.15) in Equation (4.14) and extract the coefficients of Z. Therefore, a system of algebraic equations is gained. The solutions of the system are reported in Table 6. Theorem 4.3 is derived from Table 6. The solutions to Equation (4.9) are listed in Table 7. The functions $\Psi_1(\theta, x)$, $\Psi_2(\theta, x)$, $\Psi_3(\theta, x)$, and $\Psi_4(\theta, x)$ that exist in

Table 7 are defined as follows

$$\begin{aligned}
\Psi_1(\theta,x) &= -\log\left(-1+\frac{1}{2}\tanh(\frac{1}{2}(x-v\log(\theta)))+\frac{\frac{1}{2}}{\tanh(\frac{1}{2}(x-v\log(\theta)))}\right),\\
\Psi_2(\theta,x) &= -\log\left(-1-\frac{1}{2}\tanh(-\frac{1}{2}(x-v\log(\theta)))+\frac{-\frac{1}{2}}{\tanh(-\frac{1}{2}(x-v\log(\theta)))}\right),\\
\Psi_3(\theta,x) &= -\log\left(-1-\frac{1}{2}\tanh(\frac{1}{2}(x-v\log(\theta)))+\frac{-\frac{1}{2}}{\tanh(\frac{1}{2}(x-v\log(\theta)))}\right),\\
\Psi_4(\theta,x) &= -\log\left(-1+\frac{1}{2}\tanh(-\frac{1}{2}(x-v\log(\theta)))+\frac{\frac{1}{2}}{\tanh(-\frac{1}{2}(x-v\log(\theta)))}\right).
\end{aligned}$$

Theorem 4.3. *The similarity solutions generated by* $\theta\frac{\partial}{\partial\theta}$ *parameterize either center manifolds or manifolds that contain only one function. These functions correspond to the stable manifold or center manifold.*

Proof. Eigenfunction (4.6) contains the terms $\sqrt{A'_2}$, $\sqrt{A'_2+2}$, and $\sqrt{-\lambda A'_2}$. From this, we can conclude that $A'_2 \geq 0$ and $\lambda \leq 0$. Also, in eigenfunction (4.8), there exist $\sqrt{A'_4}$, $\sqrt{A'_4+2}$, and $\sqrt{-\lambda A'_4}$, then $A'_4 \geq 0$ and $\lambda \leq 0$. □

TABLE 6. The solutions of the produced system by extracting coefficients Z.

Case	a_0	a_1	b_1	λ	c	v
6-1	0	0	0	arbitrary	arbitrary	arbitrary
6-2	−2	0	0	arbitrary	arbitrary	arbitrary
6-3	−1	0	1	0	1	arbitrary
6-4	−1	0	−1	0	−1	arbitrary
6-5	−1	0	−1	0	1	arbitrary
6-6	−1	0	1	0	−1	arbitrary
6-7	−1	0	1	arbitrary	1	0
6-8	−1	0	−1	arbitrary	−1	0
6-9	−1	0	−1	arbitrary	1	0
6-10	−1	0	1	arbitrary	−1	0
6-11	−1	1	0	0	1	arbitrary
6-12	−1	−1	0	0	−1	arbitrary
6-13	−1	−1	0	0	1	arbitrary
6-14	−1	1	0	0	−1	arbitrary
6-15	−1	1	0	arbitrary	1	0
6-16	−1	−1	0	arbitrary	−1	0
6-17	−1	−1	0	arbitrary	1	0
6-18	−1	1	0	arbitrary	−1	0
6-19	−1	1/2	1/2	0	1/2	arbitrary
6-20	−1	−1/2	−1/2	0	−1/2	arbitrary
6-21	−1	−1/2	−1/2	0	1/2	arbitrary
6-22	−1	1/2	1/2	0	−1/2	arbitrary
6-23	−1	−1/2	−1/2	arbitrary	1/2	0
6-24	−1	1/2	1/2	arbitrary	−1/2	0
6-25	−1	1/2	1/2	arbitrary	1/2	0
6-26	−1	−1/2	−1/2	arbitrary	−1/2	0

TABLE 7. The solutions of Equation (4.9).

Case	Eigenvalue	Type of invariant manifold	$Q(\theta, x)$
7-1	0	center	$-\log\left(-1+\frac{1}{\tanh(x-v\log(\theta))}\right)$
7-2	0	center	$-\log\left(-1+\frac{-1}{\tanh(-x+v\log(\theta))}\right)$
7-3	0	center	$-\log\left(-1+\frac{-1}{\tanh(x-v\log(\theta))}\right)$
7-4	0	center	$-\log\left(-1+\frac{1}{\tanh(-x+\log(\theta))}\right)$
7-5	arbitrary	stable or center	$-\log\left(-1+\frac{1}{\tanh(x)}\right)$
7-6	arbitrary	stable or center	$-\log\left(-1+\frac{-1}{\tanh(-x)}\right)$
7-7	arbitrary	stable or center	$-\log\left(-1+\frac{-1}{\tanh(x)}\right)$
7-8	arbitrary	stable or center	$-\log\left(-1+\frac{1}{\tanh(-x)}\right)$
7-9	0	center	$-\log\left(-1+\tanh(x-v\log(\theta))\right)$
7-10	0	center	$-\log\left(-1-\tanh(-x+v\log(\theta))\right)$
7-11	0	center	$-\log\left(-1-\tanh(x-v\log(\theta))\right)$
7-12	0	center	$-\log\left(-1+\tanh(-x+v\log(\theta))\right)$
7-13	arbitrary	stable or center	$-\log\left(-1+\tanh(x)\right)$
7-14	arbitrary	stable or centerr	$-\log\left(-1-\tanh(-x)\right)$
7-15	arbitrary	stable or center	$-\log\left(-1-\tanh(x)\right)$
7-16	arbitrary	stable or center	$-\log\left(-1+\tanh(-x)\right)$
7-17	0	center	$\Psi_1(\theta, x)$
7-18	0	center	$\Psi_2(\theta, x)$
7-19	0	center	$\Psi_3(\theta, x)$
7-20	0	center	$\Psi_4(\theta, x)$
7-21	arbitrary	stable or center	$-\log\left(-1-\frac{1}{2}\tanh(\frac{1}{2}x)+\frac{-\frac{1}{2}}{\tanh(\frac{1}{2}x)}\right)$
7-22	arbitrary	stable or center	$-\log\left(-1+\frac{1}{2}\tanh(-\frac{1}{2}x)+\frac{\frac{1}{2}}{\tanh(-\frac{1}{2}x)}\right)$
7-23	arbitrary	stable or center	$-\log\left(-1+\frac{1}{2}\tanh(\frac{1}{2}x)+\frac{\frac{1}{2}}{\tanh(\frac{1}{2}x)}\right)$
7-24	arbitrary	stable or center	$-\log\left(-1-\frac{1}{2}\tanh(-\frac{1}{2}x)+\frac{-\frac{1}{2}}{\tanh(-\frac{1}{2}x)}\right)$

Symmetry reduction with $\theta\frac{\partial}{\partial\theta}+\alpha\frac{\partial}{\partial x}$. Based on the calculations, the characteristic equations for $\theta\frac{\partial}{\partial\theta}+\alpha\frac{\partial}{\partial x}$ are $\frac{d\theta}{\theta}=\frac{dx}{\alpha}=\frac{d\Psi}{0}$. Then the similarity variable is $y=\theta^{\alpha}\exp(-x)$. By substituting the obtained similarity variable in Equation (4.9), a new Equation is gained

$$\frac{1}{2}y^2\frac{d^2\Psi}{dy^2}+(\frac{1}{2}-\lambda\alpha)y\frac{d\Psi}{dy}+\exp(-\Psi)+\frac{1}{2}\exp(-2\Psi)=0\cdot \tag{4.16}$$

Vector field $y\frac{\partial}{\partial y}$ is an infinitesimal generator for Equation (4.16), and $z=\log(y)$ is the similarity variable. By means of the similarity variable $z=\log(y)$, Equation (4.16) is reduced to

$$\frac{1}{2}\frac{d^2\Psi}{dz^2}-\lambda\alpha\frac{d\Psi}{dz}+\exp(-\Psi)+\frac{1}{2}\exp(-2\Psi)=0\cdot \tag{4.17}$$

We introduce the change of variable $T=\tanh(z)$, and then Equation (4.17) is reduced to

$$\frac{1}{2}(1-T^2)^2\frac{d^2\Psi}{dT^2}+(-T-\lambda\alpha)(1-T^2)\frac{d\Psi}{dT}+\exp(-\Psi)+\frac{1}{2}\exp(-2\Psi)=0\cdot \tag{4.18}$$

Suppose $S(T)=\sum_{i=0}^{K}a_iT^i+\sum_{i=1}^{K}b_iT^{-i}$ and $-\log(S(T))$ is the solution of Equation (4.18). Therefore, Equation (4.18) is reduced to

$$\frac{1}{2}(1-T^2)^2\left(\frac{-S''S+S'^2}{S^2}\right)+(-T-\lambda\alpha)(1-T^2)\left(\frac{-S'}{S}\right)+S+\frac{1}{2}S^2=0\cdot \tag{4.19}$$

To determine the value of K in Equation (4.19), we balance the highest available order of the derivative and the highest power of Ψ and find that $K = 1$. Consequently,

$$S(T) = -\log\left(a_0 + a_1 T + \frac{b_1}{T}\right), \tag{4.20}$$

is a solution of Equation (4.19). By substituting the solution (4.20) in Equation (4.19) and extracting the coefficients of T, a system of algebraic equations is gained. The solutions of the system are reported in Table 8. For solving Equation (4.9), we substitute the values of Table 8 in

$$\Psi(\theta, x) = -\log\left(a_0 + a_1 \tanh(\log(\theta^\alpha) - x) + \frac{b_1}{\tanh(\log(\theta^\alpha) - x)}\right).$$

These solutions are reported in Table 8.

Remark 4.4. The similarity solutions generated by $\theta\dfrac{\partial}{\partial\theta} + \alpha\dfrac{\partial}{\partial x}$ parameterize either center manifolds or manifolds that contain only one function. These functions correspond to either stable manifolds or center manifolds.

TABLE 8. Left: The solutions of the produced system by extracting coefficients T. Right: The solutions of Equation (4.9).

Case	a_0	a_1	b_1	λ
8-1	0	0	0	arbitrary
8-2	-2	0	0	arbitrary
8-3	-1	0	1	0
8-4	-1	0	-1	0
8-5	-1	1	0	0
8-6	-1	-1	0	0

Case	Eigenvalue	Type of invariant manifold	$U(\theta, x)$
9-1	0	center	$-\log\left(-1 + \frac{1}{\tanh(\log(\theta^\alpha) - x)}\right)$
9-2	0	center	$-\log\left(-1 - \frac{1}{\tanh(\log(\theta^\alpha) - x)}\right)$
9-3	0	center	$-\log\left(-1 + \tanh(\log(\theta^\alpha) - x)\right)$
9-4	0	center	$-\log\left(-1 - \tanh(\log(\theta^\alpha) - x)\right)$

We show some examples of center manifolds in case 7-4 and 9-2, which are listed in Tables 7 and 8 and depicted in Figure 6. Three center manifolds in case 7-19 in Table 7 are shown in Figure 7. Also, three invariant manifolds in cases 7-5, 7-7, and 7-8 in Table 7 (center or stable manifolds) are shown in Figure 7. Functions 7-7 and 7-8 have the same graph.

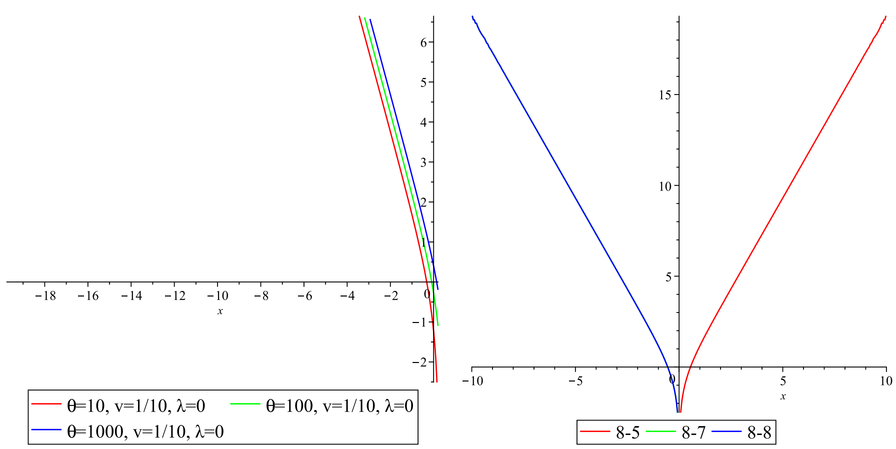

FIGURE 7. Left: Three functions of the center manifold of case 7-19 with $\lambda = 0$ in Table 7. Right: Three functions 7-5, 7-7, and 7-8 in Table 7 as the stable or center manifold.

5. Conclusion

Our intention was to develop an exact method for computing the invariant manifolds associated with evolution equations. To parameterize these manifolds, we utilized an analytical method that allowed us to derive exact solutions for the underlying equations. Our method is a combination of the parametrization method and Lie symmetry analysis method. The invariant manifolds we obtained were associated with the equilibrium solutions. To apply our method, we first computed the equilibrium solutions of the underlying equations. The next step involved modeling the eigenvalue problem using an equilibrium solution and solving it using Maple software. Eigenfunctions and eigenvalues were obtained by solving the eigenvalue problem. We modelled the invariance equation using the obtained eigenvalues. By employing tanh method, we solved invariance equations. We presented two examples to showcase the effectiveness and efficiency of the method used in computing the invariant manifolds. These examples were Fisher equation and Brain Tumor growth differential equation.

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