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Boundary controller design for stabilization of stochastic nonlinear reaction-diffusion systems with time-varying delays

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Abstract

This paper is focused on studying the stabilization problems of stochastic nonlinear reaction-diffusion systems (SNRDSs) with time-varying delays via boundary control. Firstly, the boundary controller was designed to stabilization for SNRDSs. By utilizing the Lyapunov functional method, Ito's differential formula, Wirtinger's inequality, Gronwall inequality, and LMIs, sufficient conditions are derived to guarantee the finite-time stability (FTS) of proposed systems. Secondly, the basic expressions of the control gain matrices are designed for the boundary controller. Finally, numerical examples are presented to verify the efficiency and superiority of the proposed stabilization criterion.

Keywords. Stochastic nonlinear systems, Reaction-diffusion terms, Boundary control, Time-varying delays.

2010 Mathematics Subject Classification. 93C10, 93E03, 93E15.

1. Introduction

Recently, reaction-diffusion systems (RDSs) have been extensively studied by many authors because they can be adeptly applied to the wide range of fields, including secure communication [13, 23], chemical reaction process [22], oncolytic M1-virotherapy model [7], virus transmission [16], and food web model [31]. The network structure and nonlinear dynamic behavior may both change during the movement. However, because of the spatially inhomogeneous environment, diffusion effects are usually unavoidable. Ordinary differential systems are insufficient to accurately describe them in this case. As a result, partial differential systems with reaction-diffusion terms have received a lot of attention [1, 5, 18, 20, 21, 24, 32, 35, 38].

In practical systems including nonlinear circuits, biological systems, power systems, chemical industry systems, and reaction-diffusion systems, time delays are mostly unavoidable. Oscillation or instability in RDSs can be characterized by the presence of time delay. The Lyapunov-Krasovskii function is useful when dealing with time-delay terms in RDSs. The work [25] used the Lyapunov- Krasovskii functional technique to deal with the effect of time delay on the RDSs. The work [33] assured the stability of the RDSs with time delay by employing the Lyapunov-Krasovskii function. The Lyapunov-Krasovskii function was also used to solve the stabilization problems for RDSs with time-varying delays in the study [6, 27]. As a result, time-varying delays must be considered for RDSs [2–4, 9, 12, 14, 15, 26, 28–30, 34, 39, 40].

Boundary controllers, as a particular control technique for RDSs, can be offered to achieve the required performance of RDSs while also saving costs and being easy to implement [8, 11, 17]. The back stepping method has been used to explore boundary control for RDSs. In [10], the author designed the boundary controller for finite-time stabilization of stochastic RDSs with Markovian switching and without delay. In [37], the author investigated the stabilization for RDSs with time-delay via boundary control. The authors of [19, 36] devised the Lyapunov functional technique for dealing with finite-time stabilization problems in RDSs using a boundary controller. To the best of our knowledge, few authors little attention to the stabilization problems of SNRDSs with time-varying delays via boundary control.

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Inspired by the above discussions, we investigated the FTS and stabilization for SNRDSs with time-varying delays via boundary control. The following are the main contributions of this paper: (i) A boundary controller was designed for FTS and stabilization for SNRDSs with time-varying delays. (ii) Our theoretical results reflects the effects of the boundary controller and reaction-diffusion terms on the FTS. (iii) Sufficient conditions are presented in LMIs that can be verified by Matlab LMI toolbox.

Notations: $\mathbb{R}$ – set of all real numbers; $\mathbb{R}_+$ – set of all positive real numbers; $\mathbb{R}^n$ – Euclidean space of n-dimensions; $\mathbb{R}^{m\times n}$ – Euclidean space of $(m\times n)$-dimensions; $A<0$ – real symmetric negative definite matrix; $A>0$ – real symmetric positive definite matrix; A^T – transpose of the matrix A; $\lambda_{\min}(A)$ – minimum eigen value of A; $\lambda_{\max}(A)$ – maximum eigenvalue of A; $*$ – the entries are implied by symmetric; $He\{A\}=(A+A^T)$; $\|\cdot\|$ – Euclidean norms; $\mathbb{E}(X)$ – mathematical expectation of X; $\mathcal{W}^{1,2}([0,\Omega];\mathbb{R}^n)$ – Soblev n-dimensional space of continuous functions; $\int_0^1 \Im^T(x,t)\Im(x,t)dx = \|\Im(x,t)\|^2$.

2. System Description and Preliminaries

Consider the following stochastic nonlinear reaction-diffusion systems (SNRDSs) with time-varying delays

$$\begin{aligned} d\Im(x,t) = \ & \Big[\mathcal{D}\frac{\partial^2\Im(x,t)}{\partial x^2} + \mathcal{A}\Im(x,t) + \mathcal{B}\Im(x,t-\tau(t)) + f(t,\Im(x,t)) \\ & + g(t,\Im(x,t-\tau(t)))\Big]dt + \sigma(t,\Im(x,t),\Im(x,t-\tau(t)))d\omega(t), \end{aligned} \tag{2.1}$$

with initial and Neumann boundary conditions as follows:

$$\Im(x,s) = \phi(x,s),\ x\in(0,1),\ s\in[-\tau,0], \tag{2.2}$$

$$\frac{\partial\Im(x,t)}{\partial x}\Big|_{x=0} = 0,\ \frac{\partial\Im(x,t)}{\partial x}\Big|_{x=1} = u(t), \tag{2.3}$$

where $\Im(x,t) = [\Im_1(x,t),\Im_2(x,t),...,\Im_n(x,t)]^T\in\mathbb{R}^n$ is a state vector; $t>0$ is a time variable; $x\in(0,1)$ is a space variable. $\phi(x,s) = [\phi_1(x,s),\phi_2(x,s),...,\phi_n(x,s)]^T\in\mathbb{R}^n$ is the continuous initial functions. $u(t) = [u_1(t),u_2(t),...,u_n(t)]^T\in\mathbb{R}^n$ is the boundary control input vector to be designed later. $\mathcal{D}$ is a positive definite diffusion matrix. $f,g:\mathbb{R}_+\times\mathbb{R}^n\to\mathbb{R}^n$ and $\sigma:\mathbb{R}_+\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^{n\times m}$ are the continuous nonlinear functions. $\omega(t) = [\omega_1(t),\omega_2(t),...,\omega_n(t)]^T\in\mathbb{R}^m$ is a m-dimensional Brownian motion. $\tau(t)$ are time-varying delays and satisfying the conditions $0\le\tau(t)\le\tau$ and $\dot\tau(t)\le\rho<1$. $\mathcal{A}$ and $\mathcal{B}$ are constant matrices with compatible dimensions.

Assumption 2.1. There exist nonnegative constants α_1 and α_2 such that

$$\begin{aligned} (f(l)-f(m))^T(f(l)-f(m)) \le & \alpha_1(l-m)^T(l-m), \\ (g(l)-g(m))^T(g(l)-g(m)) \le & \alpha_2(l-m)^T(l-m),\quad \forall\ l,m\in\mathbb{R}^n. \end{aligned}$$

Assumption 2.2. There exist nonnegative constants β_1 and β_2 such that

$$\begin{aligned} & trace[(\sigma(l_1,l_2)-\sigma(m_1,m_2))^T(\sigma(l_1,l_2)-\sigma(m_1,m_2))] \\ & \qquad \le \beta_1(l_1-m_1)^T(l_1-m_1) + \beta_2(l_2-m_2)^T(l_2-m_2),\quad \forall\ l_1,l_2,m_1,m_2\in\mathbb{R}^n. \end{aligned}$$

Lemma 2.3. [38] The following matrix inequality applies to any real matrices $\mathcal{M}$ and $\mathcal{N}$ and a positive definite matrix $\mathcal{S}$:

$$\mathcal{M}^T\mathcal{N} + \mathcal{N}^T\mathcal{M} \le \mathcal{M}^T\mathcal{S}^{-1}\mathcal{M} + \mathcal{N}^T\mathcal{S}\mathcal{N}.$$

Lemma 2.4. [37] For a matrix $\mathcal{M}>0$ and a state vector $y(t)\in\mathcal{W}^{1,2}([0,\Omega];\mathbb{R}^n)$ with $y(0)=0$ or $y(\Omega)=0$, we have

$$\int_0^\Omega y^T(s)My(s)ds \le \frac{4\Omega^2}{\pi^2}\int_0^\Omega \Big(\frac{dy(s)}{ds}\Big)^T M\Big(\frac{dy(s)}{ds}\Big)ds.$$

Lemma 2.5. [24] The following inequality applies for any symmetric matrix $N > 0$, any scalars a and b with $a < b$, and vector function $y(t) : [a, b] \to \mathbb{R}^n$ such that the following integral is properly defined:

$$\Big[\int_a^b y(s)ds\Big]^T N\Big[\int_a^b y(s)ds\Big] \leq (b-a)\int_a^b y^T(s)Ny(s)ds.$$

Lemma 2.6. [3] Let $\rho \in \mathbb{R}$ and $\kappa \in \mathbb{R}_+$ be a constants. If there is a function $x(t)$ that meets the criteria,

$$x(t) \leq \rho + \int_b^t \kappa x(s)ds, \quad b \leq t \leq c,$$

then one has satisfying

$$x(t) \leq \rho e^{\kappa(t-b)}.$$

Lemma 2.7. [30] Let $\Omega_1, \Omega_2, \Omega_3$ be given matrices such that $\Omega_1^T = \Omega_1$ and $\Omega_2^T = \Omega_2 > 0$. Then

$$\Omega_1 + \Omega_3^T \Omega_2^{-1}\Omega_3 < 0 \Leftrightarrow \begin{bmatrix} \Omega_1 & \Omega_3^T \\ * & -\Omega_2 \end{bmatrix} < 0 \;\; or \;\; \begin{bmatrix} -\Omega_2 & \Omega_3 \\ * & \Omega_1 \end{bmatrix} < 0.$$

Definition 2.8. [10] Given three constants z_1, z_2 and $\mathcal{T}$ with $z_1 < z_2$, the SNRDSs (2.1) is called finite-time stable (FTS) with respect to $(z_1, z_2, \mathcal{T})$ if for given initial conditions satisfying

$$\mathbb{E}\Big\{\sup_{s\in[-\tau,0]} ||\Im(x,s)||^2\Big\} \leq z_1 \Rightarrow \mathbb{E}||\Im(x,t)||^2 < z_2, \forall\, t \in [0, \mathcal{T}].$$

Definition 2.9. [38] The SNRDSs (2.1) is said to be stabilizable if there exist control gain matrices for boundary controller, such that the SNRDSs (2.1) FTS with respect to $(z_1, z_2, \mathcal{T})$.

3. Main Results

In this section, we investigated the FTS and stabilization for SNRDSs (2.1) via boundary control. The boundary controller is designed as

$$u(t) = \Theta \int_0^1 \Im(x,t)dx, \tag{3.1}$$

where Θ is a control gain matrix will be designed later.

Theorem 3.1. Under Assumptions (2.1)-(2.2), the SNRDSs (2.1) is said to be FTS with respect to given constants $(z_1, z_2, \mathcal{T})$ if there exist constant $\kappa > 0$, symmetric positive definite matrices $\mathcal{P}, \mathcal{Q}, \mathcal{R}, \mathcal{S}_1, \mathcal{S}_2$ such that the following LMIs holds:

$$(i)\;\; \Pi = \begin{bmatrix} \Pi_{11} & \Pi_{12} & \Pi_{13} & \Pi_{14} \\ * & \Pi_{22} & \Pi_{23} & \Pi_{24} \\ * & * & \Pi_{33} & \Pi_{34} \\ * & * & * & \Pi_{44} \end{bmatrix} < 0, \tag{3.2}$$

$$(ii)\;\; \frac{z_1}{\lambda_{\min}(\mathcal{P})} e^{\kappa\mathcal{T}}\Big[\lambda_{\max}(\mathcal{P}) + \tau e^{\kappa\tau}\lambda_{\max}(\mathcal{Q}) + \tau^2 e^{\kappa\tau}\lambda_{\max}(\mathcal{R})\Big] < z_2, \tag{3.3}$$

where

$$\Pi_{11} = He(\mathcal{P}\mathcal{A} + \mathcal{D}\mathcal{P}\Theta) + \mathcal{Q} + \tau\mathcal{R} + \mathcal{P}\mathcal{S}_1^{-1}\mathcal{P} + \alpha_1\mathcal{S}_1 + \mathcal{P}\mathcal{S}_2^{-1}\mathcal{P} + \lambda_{\max}(\mathcal{P})\beta_1 - \kappa\mathcal{P},\; \Pi_{12} = -\Theta^T\mathcal{P}^T\mathcal{D}^T,$$

$$\Pi_{13} = \mathcal{P}\mathcal{B},\; \Pi_{14} = 0, \Pi_{22} = -\frac{1}{2}\pi^2\mathcal{P}\mathcal{D},\; \Pi_{23} = 0,\; \Pi_{24} = 0,\; \Pi_{33} = -(1-\rho)e^{\kappa\tau}\mathcal{Q} + \alpha_2\mathcal{S}_2 + \lambda_{\max}(\mathcal{P})\beta_2,$$

$$\Pi_{34} = 0,\; \Pi_{44} = -\frac{1}{\tau}e^{\kappa\tau}\mathcal{R}.$$

Proof. Construct the following Lyapunov-Krasovskii functional candidate:

$$V(t, \Im(x,t)) = \sum_{p=1}^{3} V_p(t, \Im(x,t)), \tag{3.4}$$

where

$$\begin{aligned}
V_1(t, \Im(x,t)) &= \int_0^1 \Im^T(x,t)\mathcal{P}\Im(x,t)dx, \\
V_2(t, \Im(x,t)) &= \int_0^1 \int_{t-\tau(t)}^{t} e^{\kappa(t-s)}\Im^T(x,s)\mathcal{Q}\Im(x,s)dsdx, \\
V_3(t, \Im(x,t)) &= \int_0^1 \int_{-\tau(t)}^{0} \int_{t+\theta}^{t} e^{\kappa(t-s)}\Im^T(x,s)\mathcal{R}\Im(x,s)dsd\theta dx.
\end{aligned}$$

Calculating the derivative of $V(t, \Im(x,t))$ along the trajectories of SNRDSs (2.1) by Ito's differential formula, we obtain that

$$\mathcal{L}V(t, \Im(x,t)) = \mathcal{L}V_1(t, \Im(x,t)) + \mathcal{L}V_2(t, \Im(x,t)) + \mathcal{L}V_3(t, \Im(x,t)). \tag{3.5}$$

Further, we have

$$\begin{aligned}
\mathcal{L}V_1(t, \Im(x,t)) =& 2\int_0^1 \Im^T(x,t)\mathcal{P}\Big[\mathcal{D}\frac{\partial^2 \Im(x,t)}{\partial x^2} + \mathcal{A}\Im(x,t) + \mathcal{B}\Im(x,t-\tau(t)) + f(t,\Im(x,t)) \\
&+ g(t, \Im(x,t-\tau(t)))\Big]dx + \int_0^1 trace\Big[\sigma^T(t)\mathcal{P}\sigma(t)\Big]dx \\
&- \kappa\int_0^1 \Im^T(x,t)\mathcal{P}\Im(x,t)dx + \kappa V_1(t, \Im(x,t)),
\end{aligned} \tag{3.6}$$

where $\sigma(t) = \sigma(t, \Im(x,t), \Im(x,t-\tau(t)))$.

$$\begin{aligned}
\mathcal{L}V_2(t, \Im(x,t)) =& \kappa\int_0^1 \int_{t-\tau(t)}^{t} e^{\kappa(t-s)}\Im^T(x,s)\mathcal{Q}\Im(x,s)dsdx + \int_0^1 \Im^T(x,t)\mathcal{Q}\Im(x,t)dx \\
&- (1-\dot{\tau}(t))e^{\kappa\tau(t)}\int_0^1 \Im^T(x,t-\tau(t))\mathcal{Q}\Im(x,t-\tau(t))dx \\
\leq& \kappa V_2(t, \Im(x,t)) + \int_0^1 \Im^T(x,t)\mathcal{Q}\Im(x,t)dx \\
&- (1-\rho)e^{\kappa\tau}\int_0^1 \Im^T(x,t-\tau(t))\mathcal{Q}\Im(x,t-\tau(t))dx,
\end{aligned} \tag{3.7}$$

$$\begin{aligned}
\mathcal{L}V_3(t, \Im(x,t)) =& \kappa\int_0^1 \int_{-\tau(t)}^{0} \int_{t+\theta}^{t} e^{\kappa(t-s)}\Im^T(x,s)\mathcal{R}\Im(x,s)dsd\theta dx \\
&+ \tau\int_0^1 \Im^T(x,t)\mathcal{R}\Im(x,t)dx - e^{\kappa\tau(t))}\int_0^1 \int_{t-\tau(t)}^{t} \Im^T(x,s)\mathcal{R}\Im(x,s)dsdx \\
\leq& \kappa V_3(t, \Im(x,t)) + \tau\int_0^1 \Im^T(x,t)\mathcal{R}\Im(x,t)dx \\
&- e^{\kappa\tau}\int_0^1 \int_{t-\tau(t)}^{t} \Im^T(x,s)\mathcal{R}\Im(x,s)dsdx.
\end{aligned} \tag{3.8}$$

According to Lemma 2.3 and Assumption 2.1, we have

$$\begin{aligned} 2\Im^T(x,t)\mathcal{P}f(t,\Im(x,t)) \leq& \Im^T(x,t)\mathcal{P}\mathcal{S}_1^{-1}\mathcal{P}\Im(x,t) + f^T(t,\Im(x,t))\mathcal{S}_1 f(t,\Im(x,t)) \\ \leq& \Im^T(x,t)\mathcal{P}\mathcal{S}_1^{-1}\mathcal{P}\Im(x,t) + \Im^T(x,t)\alpha_1\mathcal{S}_1\Im(x,t), \end{aligned} \tag{3.9}$$

similarly

$$2\Im^T(x,t)\mathcal{P}g(t,\Im(x,t-\tau(t))) \leq \Im^T(x,t)\mathcal{P}\mathcal{S}_2^{-1}\mathcal{P}\Im(x,t) + \Im^T(x,t-\tau(t))\alpha_2\mathcal{S}_2\Im(x,t-\tau(t)). \tag{3.10}$$

Based on Assumption 2.2, we have

$$trace\Big[\sigma^T(t)\mathcal{P}\sigma(t)\Big] \leq \lambda_{\max}(\mathcal{P})\Big[\Im^T(x,t)\beta_1\Im(x,t) + \Im^T(x,t-\tau(t))\beta_2\Im(x,t-\tau(t))\Big]. \tag{3.11}$$

By using integration by parts and Neumann boundary condition (2.3), we obtain that

$$\begin{aligned} \int_0^1 \Im^T(x,t)\mathcal{D}\frac{\partial^2\Im(x,t)}{\partial x^2}dx =& \Big[\Im^T(x,t)\mathcal{D}\frac{\partial\Im(x,t)}{\partial x}\Big]_{x=0}^{x=1} - \int_0^1 \frac{\partial\Im^T(x,t)}{\partial x}\mathcal{D}\frac{\partial\Im(x,t)}{\partial x}dx \\ =& \int_0^1 \Im^T(1,t)\mathcal{D}\Theta\Im(x,t)dx - \int_0^1 \frac{\partial\Im^T(x,t)}{\partial x}\mathcal{D}\frac{\partial\Im(x,t)}{\partial x}dx. \end{aligned} \tag{3.12}$$

To get $\bar{\Im}(x,t)=0$, for introduce a state variable $\bar{\Im}(x,t)=\Im(x,t)-\Im(1,t)$, and the following inequality is holds:

$$\frac{\partial\Im^T(x,t)}{\partial x}\mathcal{D}\frac{\partial\Im(x,t)}{\partial x} = \frac{\partial\bar{\Im}^T(x,t)}{\partial x}\mathcal{D}\frac{\partial\bar{\Im}(x,t)}{\partial x}. \tag{3.13}$$

Applying Lemma 2.4, we get

$$\begin{aligned} \int_0^1 \Im^T(x,t)\mathcal{D}\frac{\partial^2\Im(x,t)}{\partial x^2}dx \leq& \int_0^1 \Im^T(1,t)\mathcal{D}\Theta\Im(x,t)dx - \frac{1}{4}\pi^2\int_0^1 \bar{\Im}^T(x,t)\mathcal{D}\bar{\Im}(x,t)dx \\ \leq& \int_0^1 \Im^T(x,t)\mathcal{D}\Theta\Im(x,t)dx - \int_0^1 \bar{\Im}^T(x,t)\mathcal{D}\Theta\Im(x,t)dx \\ &- \frac{1}{4}\pi^2\int_0^1 \bar{\Im}^T(x,t)\mathcal{D}\bar{\Im}(x,t)dx. \end{aligned} \tag{3.14}$$

Based on Lemma 2.5, we have

$$-\int_{t-\tau(t)}^t \Im^T(x,s)R\Im(x,s)ds \leq -\frac{1}{\tau}\Big(\int_{t-\tau(t)}^t \Im(x,s)\Big)^T R\Big(\int_{t-\tau(t)}^t \Im(x,s)\Big). \tag{3.15}$$

Combining the inequalities (3.5)-(3.15), we have

$$\mathcal{L}V(t,\Im(x,t)) \leq \int_0^1 \varpi^T(x,t)\Pi\varpi(x,t)dx + \kappa V(t,\Im(x,t)), \tag{3.16}$$

where

$$\varpi(x,t) = \Big[\Im(x,t) \quad \bar{\Im}(x,t) \quad \Im(x,t-\tau(t)) \quad \int_{t-\tau(t)}^t \Im(x,s)ds\Big]^T.$$

Based on the inequality (3.2), we have

$$\mathcal{L}V(t,\Im(x,t)) \leq \kappa V(t,\Im(x,t)). \tag{3.17}$$

Then by taking mathematical expectation,

$$D^+\mathbb{E}V(t,\Im(x,t)) = \mathbb{E}\mathcal{L}V(t,\Im(x,t)) \leq \kappa\mathbb{E}V(t,\Im(x,t)). \tag{3.18}$$

Integrating from 0 to t and according to the Lemma 2.6, we have

$$\begin{aligned}\mathbb{E}V(t,\Im(x,t)) \leq& \mathbb{E}V(0,\Im(x,0)) + \kappa\int_0^t \mathbb{E}V(t,\Im(x,t))\\ \leq& e^{\kappa t}\Big[\mathbb{E}V(0,\Im(x,0))\Big],\ \forall\ t\in[0,\mathcal{T}].\end{aligned} \tag{3.19}$$

From (3.4), we obtain that

$$\begin{aligned}\mathbb{E}V(0,\Im(x,0)) =& \mathbb{E}\Big\{\int_0^1 \Im^T(x,0)\mathcal{P}\Im(x,0)dx + \int_0^1\int_{-\tau(t)}^0 \Im^T(x,s)\mathcal{Q}\Im(x,s)dsdx\\ &+ \int_0^1\int_{-\tau(t)}^0\int_\theta^0 e^{\kappa(t-s)}\Im^T(x,s)\mathcal{R}\Im(x,s)dsd\theta dx\Big\}\\ \leq& \Big[\lambda_{\max}(\mathcal{P}) + \tau e^{\kappa\tau}\lambda_{\max}(\mathcal{Q}) + \tau^2 e^{\kappa\tau}\lambda_{\max}(\mathcal{R})\Big]\mathbb{E}\Big\{\sup_{s\in[-\tau,0]}\|\Im(x,s)\|^2\Big\}.\end{aligned} \tag{3.20}$$

Also, we have

$$\mathbb{E}V(t,\Im(x,t)) \geq \lambda_{\min}(\mathcal{P})\mathbb{E}\Big\{\int_0^1 \Im^T(x,t)\Im(x,t)dx\Big\} = \lambda_{\min}(\mathcal{P})\mathbb{E}\|\Im(x,t)\|^2. \tag{3.21}$$

Combining the inequalities (3.19)-(3.21), we have

$$\begin{aligned}\mathbb{E}\|\Im(x,t)\|^2 \leq& \frac{e^{\kappa t}}{\lambda_{\min}(\mathcal{P})}\Big[\lambda_{\max}(\mathcal{P}) + \tau e^{\kappa\tau}\lambda_{\max}(\mathcal{Q}) + \tau^2 e^{\kappa\tau}\lambda_{\max}(\mathcal{R})\Big]\\ &\times \mathbb{E}\Big\{\sup_{s\in[-\tau,0]}\|\Im(x,s)\|^2\Big\},\ \forall\ t\in[0,\mathcal{T}].\end{aligned} \tag{3.22}$$

Considering the inequality (3.3), when the following initial condition holds:

$$\mathbb{E}\Big\{\sup_{s\in[-\tau,0]}\|\Im(x,s)\|^2\Big\} < z_1,$$

it implies immediately that $\mathbb{E}\|\Im(x,t)\|^2 < z_2,\ \forall\ t\in[0,\mathcal{T}]$. According to the Definition 2.8, the SNRDSs (2.1) is FTS with respect to given constants $(z_1, z_2, \mathcal{T})$. The proof is completed. □

The next theorem states that the control gain matrix can be designed to obtain the stabilization for SNRDSs (2.1).

Theorem 3.2. Under Assumptions 2.1-2.2, the SNRDSs (2.1) is stabilizable if there exist constant $\kappa > 0$, symmetric positive definite matrices $\mathcal{P}, \mathcal{Q}, \mathcal{R}, \mathcal{S}_1, \mathcal{S}_2$, and constant matrix $\mathcal{K}$, such that the following LMIs (3.3) and

$$(iii)\ \ \Xi = \begin{bmatrix} \Xi_{11} & \Xi_{12} & \Xi_{13} & \Xi_{14} & \Xi_{15} & \Xi_{16}\\ * & \Xi_{22} & \Xi_{23} & \Xi_{24} & \Xi_{25} & \Xi_{26}\\ * & * & \Xi_{33} & \Xi_{34} & \Xi_{35} & \Xi_{36}\\ * & * & * & \Xi_{44} & \Xi_{45} & \Xi_{46}\\ * & * & * & * & \Xi_{55} & \Xi_{56}\\ * & * & * & * & * & \Xi_{66}\end{bmatrix} < 0, \tag{3.23}$$

where

$$\begin{aligned}&\Xi_{11} = He(\mathcal{P}\mathcal{A} + \mathcal{D}\mathcal{K}) + \mathcal{Q} + \tau\mathcal{R} + \alpha_1\mathcal{S}_1 + \lambda_{\max}(\mathcal{P})\beta_1 - \kappa\mathcal{P},\ \Xi_{12} = -\mathcal{K}^T\mathcal{D}^T,\ \Xi_{13} = \mathcal{P}\mathcal{B},\ \Xi_{14} = 0,\\ &\Xi_{15} = \mathcal{P},\ \Xi_{16} = \mathcal{P},\ \Xi_{22} = -\frac{1}{2}\pi^2\mathcal{P}\mathcal{D},\ \Xi_{23} = 0,\ \Xi_{24} = 0,\ \Xi_{25} = 0,\ \Xi_{26} = 0,\ \Xi_{33} = -(1-\rho)e^{\kappa\tau}\mathcal{Q}\\ &+\alpha_2\mathcal{S}_2 + \lambda_{\max}(\mathcal{P})\beta_2,\ \Xi_{34} = 0,\ \ \Xi_{35} = 0,\ \Xi_{36} = 0,\ \Xi_{44} = -\frac{1}{\tau}e^{\kappa\tau}\mathcal{R},\ \Xi_{45} = 0,\ \Xi_{46} = 0,\ \Xi_{55} = -\mathcal{S}_1,\\ &\Xi_{56} = 0,\ \Xi_{66} = -\mathcal{S}_2,\end{aligned}$$

are satisfied. Moreover, the control gain matrix is designed by

$$(iv) \quad \Theta = \mathcal{K}\mathcal{P}^{-1}. \tag{3.24}$$

Proof. Clearly, the proof of the Theorem 3.2 follows from Lemma 2.7 and Theorem 3.1. □

Remark 3.3. The obtained results in this paper extended with improve the results in [10, 38]. In [38], the author discussed the exponential stability and stabilization for stochastic nonlinear systems with time-delays and exogenous disturbances via event-triggered feedback control. In [10], the author discussed the FTS and stabilization of stochastic Markovian switching RDSs using boundary control. In this paper, we discussed the FTS and stabilization of SNRDSs with time-varying delays via boundary control.

Remark 3.4. In [34], the author discussed the FTS of impulsive RDSs. In [17], the author discussed the exponential stabilization of RDSs via intermittent boundary control. In [37], the author discussed the FTS and stabilization for RDSs by using boundary control. In [36], the author discussed the stabilization of RDSs via boundary control. However, those authors have dealt with the stability and stabilization problems without stochastic perturbations. In fact, noise presented a fundamental issue in the transmission of information impacting all facets of the neuron systems operating within the neuron systems. It is worth noting that, stochastic perturbations is introduced into the reaction-diffusion systems, which may be suitable to addressing a practical situations.

Remark 3.5. In this paper, Theorem 3.2 presents a sufficient condition to guarantee the stabilization for a class of SNRDSs with time-varying delays via boundary control. In [38], the author investigated the stabilization for a class of stochastic nonlinear systems with time delays and exogenous disturbances by using event-triggered feedback control. It's an unfortunate that the reaction-diffusion terms are not taken into account. Thus, our obtained results are more extensive than those reported in [38].

Remark 3.6. From Neumann boundary condition (2.3), let the boundary control input $u(t) = 0$. Then, the Neumann boundary condition (2.3) can be rewritten as:

$$\frac{\partial \Im(x,t)}{\partial x}|_{x=0} = 0, \; \frac{\partial \Im(x,t)}{\partial x}|_{x=1} = 0. \tag{3.25}$$

The following corollary states that the FTS for SNRDSs (2.1) without boundary control.

Corollary 3.7. Under Assumptions 2.1-2.2, the SNRDSs (2.1) without boundary control is said to be FTS with respect to given constants $(z_1, z_2, \mathcal{T})$ if there exist constant $\kappa > 0$, symmetric positive definite matrices $\mathcal{P}, \mathcal{Q}, \mathcal{R}, \mathcal{S}_1, \mathcal{S}_2$ such that the following LMIs (3.3) and

$$(v) \quad \Gamma = \begin{bmatrix} \Gamma_{11} & \Gamma_{12} & \Gamma_{13} & \Gamma_{14} & \Gamma_{15} & \Gamma_{16} \\ * & \Gamma_{22} & \Gamma_{23} & \Gamma_{24} & \Gamma_{25} & \Gamma_{26} \\ * & * & \Gamma_{33} & \Gamma_{34} & \Gamma_{35} & \Gamma_{36} \\ * & * & * & \Gamma_{44} & \Gamma_{45} & \Gamma_{46} \\ * & * & * & * & \Gamma_{55} & \Gamma_{56} \\ * & * & * & * & * & \Gamma_{66} \end{bmatrix} < 0, \tag{3.26}$$

where

$$\Gamma_{11} = He(\mathcal{P}\mathcal{A}) + \mathcal{Q} + \tau\mathcal{R} + \alpha_1\mathcal{S}_1 + \lambda_{\max}(\mathcal{P})\beta_1 - \kappa\mathcal{P}, \; \Gamma_{12} = 0, \; \Gamma_{13} = \mathcal{P}\mathcal{B}, \; \Gamma_{14} = 0, \; \Gamma_{15} = \mathcal{P},$$
$$\Gamma_{16} = \mathcal{P}, \; \Gamma_{22} = -\frac{1}{2}\pi^2\mathcal{P}\mathcal{D}, \; \Gamma_{23} = 0, \; \Gamma_{24} = 0, \; \Gamma_{25} = 0, \; \Gamma_{26} = 0, \; \Gamma_{33} = -(1-\rho)e^{\kappa\tau}\mathcal{Q} + \alpha_2\mathcal{S}_2$$
$$+ \lambda_{\max}(\mathcal{P})\beta_2, \; \Gamma_{34} = 0, \; \Gamma_{35} = 0, \; \Gamma_{36} = 0, \; \Gamma_{44} = -\frac{1}{\tau}e^{\kappa\tau}\mathcal{R}, \; \Gamma_{45} = 0, \; \Gamma_{46} = 0, \; \Gamma_{55} = -\mathcal{S}_1,$$
$$\Gamma_{56} = 0, \; \Gamma_{66} = -\mathcal{S}_2,$$

are satisfied.

4. Numerical Example

In this section, numerical example are given to illustrate the our boundary controller are effective.

Consider the following 2-dimensional SNRDSs with time-varying delays:

$$\begin{cases} d\Im(x,t) = & \Big[\mathcal{D}\frac{\partial^2\Im(x,t)}{\partial x^2} + \mathcal{A}\Im(x,t) + \mathcal{B}\Im(x,t-\tau(t)) + f(t,\Im(x,t)) \\ & +g(t,\Im(x,t-\tau(t)))\Big]dt + \sigma(t,\Im(x,t),\Im(x,t-\tau(t)))d\omega(t), \end{cases} \tag{4.1}$$

where

$$\mathcal{D} = \begin{bmatrix} 0.3 & 0 \\ 0 & 0.3 \end{bmatrix},\ \mathcal{A} = \begin{bmatrix} 0.1 & -0.1 \\ 0.2 & -0.1 \end{bmatrix},\ \mathcal{B} = \begin{bmatrix} 0.2 & -1.2 \\ 0.5 & -1.0 \end{bmatrix},$$
$$f(t,\Im(x,t)) = 0.1(1+\sin(t))\Im(x,t),$$
$$g(t,\Im(x,t-\tau(t))) = 0.1(1+\cos(t))\Im(x,t-\tau(t)),$$
$$\sigma(t,\Im(x,t),\Im(x,t-\tau(t))) = 0.2\Im(x,t) + 0.2\Im(x,t-\tau(t)).$$

The initial and Neumann boundary conditions of SNRDSs (4.1) are

$$\begin{cases} \Im_1(x,s) = & 0.01(1-\sin(0.5\pi x))In(50(s-0.5)), \\ \Im_2(x,s) = & 0.01(1-\cos(0.5\pi x))In(50(s-0.5)), \end{cases}$$

and

$$\frac{\partial\Im(x,t)}{\partial x}|_{x=0} = 0,\ \frac{\partial\Im(x,t)}{\partial x}|_{x=1} = u(t).$$

The boundary controller is

$$u(t) = \Theta\int_0^1 \Im(x,t)dx. \tag{4.2}$$

To stabilize the SNRDSs (4.1), for let, $\kappa = 2.1,\ \tau = 0.5,\ \rho = 0.3,\ z_1 = 1,\ z_2 = 5$, and $\mathcal{T} = 10$. Solve the LMIs in Theorem 3.2 by Matlab LMI toolbox, we get

$$\mathcal{P} = \begin{bmatrix} 0.2331 & 0.0259 \\ 0.0259 & 0.1610 \end{bmatrix},\ \mathcal{Q} = \begin{bmatrix} 0.3117 & 0.0002 \\ 0.0002 & 0.3097 \end{bmatrix},\ \mathcal{R} = \begin{bmatrix} 0.0161 & 0.0000 \\ 0.0000 & 0.0161 \end{bmatrix},$$
$$\mathcal{S}_1 = \begin{bmatrix} 0.4949 & -0.0087 \\ -0.0087 & 0.5195 \end{bmatrix},\ \mathcal{S}_2 = \begin{bmatrix} 0.4938 & -0.0087 \\ -0.0087 & 0.5186 \end{bmatrix},\ \mathcal{K} = \begin{bmatrix} -1.0105 & 0.0253 \\ 0.0253 & -1.1032 \end{bmatrix}.$$

Furthermore, the corresponding control gain matrix obtained as follows:

$$\Theta = \mathcal{K}P^{-1} = \begin{bmatrix} -4.4312 & 0.8713 \\ 0.8869 & -6.9933 \end{bmatrix}.$$

Therefore, based on Theorem 3.2, the SNRDSs (4.1) is FTS with respect to given constants $(z_1, z_2, \mathcal{T})$. Under boundary controller (4.2), the system states $\Im_q(x,t)(q = 1,2)$ are shown in Figure 2, and we see that they achieve finite-time stabilization for SNRDSs (4.1).

To prove the efficiency of boundary controller, for let $u(t) = 0$, i.e., SNRDSs (4.1) without boundary controller. Then, Fig.1 are displays the system states $\Im_q(x,t)(q = 1,2)$, which means that, SNRDSs (4.1) without boundary controller are does not realize the FTS. This illustrates that the our designed boundary controller are effective.

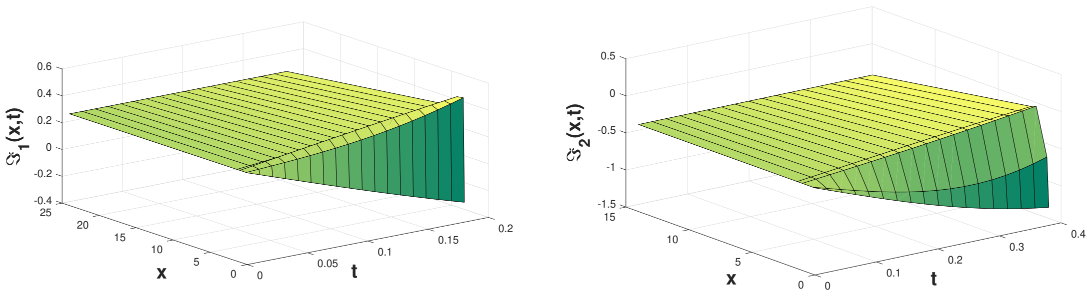

FIGURE 1. Trajectories of SNRDSs (4.1) without boundary control.

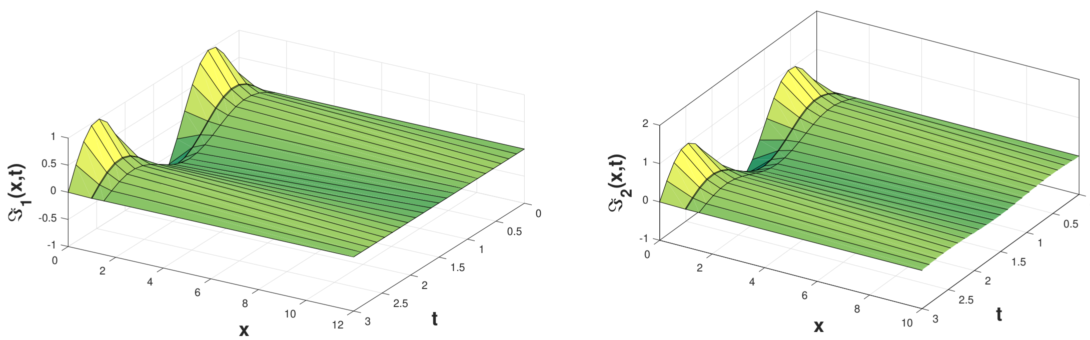

FIGURE 2. Trajectories of SNRDSs (4.1) with boundary control.

5. Conclusion

In this paper, boundary controller design for the stabilization of SNRDSs with time-varying delays is discussed. Both SNRDSs with and without the boundary controller are discussed. By constructing the Lyapunov-Krasovskii function, using Ito's differential formula, Wirtinger's inequality, Gronwall inequality, and LMIs, sufficient conditions are derived to ensure that the FTS of SNRDSs. Furthermore, the control gain matrices are designed for the boundary controller with delay-dependent results for the stabilization of proposed systems. At last, numerical examples are given to show the efficiency and superiority of obtained theoretical results. Our future study will focus on the stabilization problems for fractional-order SNRDSs using boundary control.

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