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Application of Kudryashov and functional variable method s to solve the complex KdV equation

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Abstract In this present work, the Kudryashov method and the functional variable method are used to construct exact solutions of the complex Korteweg-de Vries (KdV) equation. The Kudryashov method and the functional variable method are powerful methods for obtaining exact solutions of nonlinear evolution equations.

Keywords. Kudryashov method, functional variable method, complex Korteweg-de Vries equation.
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1. Introduction

It is well known that nonlinear complex physical phenomena are related to nonlinear evolution equations (NLEEs) which are involved in many fields from physics, biology, chemistry, mechanics, etc. As mathematical models of the phenomena, the investigation of exact solutions of NLEEs will help us to understand these phenomena better. Many effective methods for obtaining exact solutions of NLEEs have been established and developed, such as the solitary wave ansatz method [1, 2, 3], the first integral method [4, 5, 6], Jacobi elliptic function method [7, 8, 9], F-expansion method [10, 11, 12], the functional variable method [13, 14, 15], modification of truncated expansion method [16, 17, 18, 19, 20, 21] and so on.
The aim of this paper is to construct exact solutions of the complex KdV equation by using the Kudryashov method and the functional variable method.
Since the 80s of last century, the coupled KdV equations as an important mathematical model has been studied widely. In 1981, Fuchssteiner [22] made a detailed study of four coupled KdV equations and gave the bi-Hamiltonian structure of them. One coupled set of KdV equations is the complex-coupled KdV equations

$$u_t = u_{xxx} + 6uu_x + 6\phi\phi_x, \tag{1.1}$$

$$\phi_t = \phi_{xxx} + 6u\phi_x + 6u_x\phi. \tag{1.2}$$

The integrability of the equations was discussed by the bi-Hamiltonian structure [22, 23] and Lax pair [24]. Later, Oevel [23] pointed out that" inserting a complex ansatz $u + i\phi$ into the KdV it is a complex version of the KdV " and the complex version of

the KdV possess two conservation laws in every order.
In this paper we consider the complex version of KdV equation

$$U_t + \mu_1 U U_x + \mu_3 U_{xxx} = 0, \tag{1.3}$$

where $U(x,t)$ is a complex-valued function of the spatial variable x and the temporal variable t, μ_1 and μ_3 are real constants. Eq. (1.3) is completely equal to Eqs. (1.1), (1.2). In fact, letting $\mu_1 = \mu_3 = 1$, substituting the $U(x,t) = p(x,t) + iq(x,t)$ into the Eq. (1.3), separating the real part and the imaginary part from Eq. (1.3), we can obtain a set of equations about $p(x,t)$ and $q(x,t)$. After we rewrite the equations under the transformation: $p(x,t) \to 6u(x,t)$, $q(x,t) \to 6i\phi(x,t)$, $x \to -x$, we can obtain Eqs. (1.1), (1.2).
This paper is organized as follows: In section 2 and 3, we describe briefly the functional variable method and the Kudryashov method. In section 4, we apply the proposed methods to solve the complex KdV equation. In section 5, Conclusions will be presented in final.

2. The functional variable method

Consider a nonlinear evolution equation

$$P(u, u_t, u_x, u_{tt}, u_{xt}, u_{xx}, \ldots) = 0, \tag{2.1}$$

where $u = u(x,t)$ is an unknown complex-valued function, P is a polynomial in $u = u(x,t)$ and its various partial derivatives, in which the highest order derivatives and nonlinear terms are involved.
First we introduce the new wave variable as combining the independent variables x and t into one variable $\xi = k(x - ct)$, we suppose that

$$u(x,t) = U(z), \; z = i\xi. \tag{2.2}$$

The travelling wave variable (2.2) permits reducing Eq. (2.1) to an ODE for $U = U(z)$

$$P(U, -ikcU', ikU', \ldots) = 0, \tag{2.3}$$

where $U' = \frac{dU}{dz}$.
Let us make a transformation in which the unknown function U is considered as a functional variable in the form

$$U_z = F(U) \tag{2.4}$$

and some successive derivatives of U are

$$\begin{aligned} U_{zz} &= \frac{1}{2}(F^2)', \\ U_{zzz} &= \frac{1}{2}(F^2)''\sqrt{F^2}, \\ U_{zzzz} &= \frac{1}{2}[(F^2)''' + (F^2)''(F^2)'], \\ &\vdots \end{aligned} \tag{2.5}$$

The ODE (2.3) can be reduced in terms of U, F and its derivatives on using the expressions of (2.5) into (2.3) gives

$$R(U, F, F', F'', F''', F^{(4)}, \dots) = 0. \tag{2.6}$$

The key idea of this particular form (2.6) is of special interest because it admits analytical solutions for a large class of nonlinear wave type equations. After integration, Eq. (2.6) provides the expression of F, and this, together with (2.4), give appropriate solutions to the original problem.

3. Modification of truncated expansion method

The main steps of the Kudryashov method are the following:

Step1. Determination of the dominant term with highest order of singularity. To find dominant terms, we substitute

$$U = z^{-p}, \tag{3.1}$$

to all terms of Eq. (2.3). Then we compare degrees of all terms of Eq. (2.3) and choose two or more with the lowest degree. The maximum value of p is the pole of Eq. (2.3) and we denote it as N. This method can be applied when N is integer. If the value N is non-integer, one can transform the equation studied.

Step2. We look for exact solution of Eq. (2.3) in the form

$$U(z) = \sum_{i=0}^{N} b_i Q^i(z), \tag{3.2}$$

where $b_i (i = 0, 1, \dots, N)$ are constants to be determined later, such that $b_N \neq 0$, while $Q(z)$ has the form

$$Q(z) = \frac{1}{1 + dexp(z)}, \tag{3.3}$$

which is a solution to the Riccati equation

$$Q'(z) = Q^2(z) - Q(z),$$

where d is arbitrary constant.

Step3. We can calculate necessary number of derivative of function U. It is easy to do using Maple or Mathematica package. Using case $N = 2$ we have some derivatives of function $U(z)$ in the form

$$\begin{aligned}
U &= b_0 + b_1 Q + b_2 Q^2, \\
U_z &= -b_1 Q + (b_1 - 2b_2) Q^2 + 2b_2 Q^3, \\
U_{zz} &= b_1 Q + (-3b_1 + 4b_2) Q^2 + (2a_1 - 10a_2) Q^3 + 6b_2 Q^4, \\
U_{zzz} &= -b_1 Q + (7b_1 - 8b_2) Q^2 + (-12b_1 + 38b_2) Q^3 + (6b_1 - 54b_2) Q^4 + 24b_2 Q^5.
\end{aligned} \tag{3.4}$$

Step4. We substitute expressions given by Eqs. (3.2)-(3.4) in Eq. (2.3). Then we collect all terms with the same powers of function $Q(z)$ and equate expressions to zero. As a result we obtain algebraic system of equations. Solving this system we get the values of unknown parameters.

4. Applications

In this section, we construct the exact travelling wave solution to Eq.(1.3) using the presented methods. Firstly, we propose a complex travelling wave solution to the complex KdV equation in the form

$$U(x,t) = U(z),\ z = ik(x-ct), \tag{4.1}$$

where k and c are real constants to be determined later.
Substituting (4.1) into (1.3), we have

$$cU' - \mu_1 UU' + k^2\mu_3 U''' = 0, \tag{4.2}$$

where $U' = \frac{dU(z)}{dz}$.
Integrating Eq.(4.2) with respect z and considering the constant of integration to be zero, we get

$$cU - \mu_1 U^2 + k^2\mu_3 U'' = 0. \tag{4.3}$$

4.1. **The functional variable method to solve the complex KdV equation.** We use the transformation

$$U_z = F(U), \tag{4.4}$$

that will convert Eq.(4.3) to

$$\frac{(F^2(U))'}{2} = \frac{-2c}{k^2\mu_3}U + \frac{\mu_1}{k^2\mu_3}U^2. \tag{4.5}$$

According to Eq.(2.5), we get from Eq (4.5) the expression of the function $F(U)$ as

$$F(U) = \sqrt{\frac{-c}{k^2\mu_3}}U\sqrt{1-\frac{\mu_1}{3c}U} \tag{4.6}$$

Using transformation (2.4), and then setting the constants of integration to zero, we can obtain the following result:

$$U(z) = -\frac{3c}{\mu_1}csch^2(\frac{1}{2}\sqrt{\frac{-c}{k^2\mu_3}}z) \tag{4.7}$$

When $\frac{c}{\mu_3} < 0$, we have the following hyperbolic solutions:

$$U_1(x,t) = -\frac{3c}{\mu_1}csch^2(\frac{1}{2}\sqrt{\frac{-c}{k^2\mu_3}}(ik(x-ct))), \tag{4.8}$$

$$U_2(x,t) = \frac{3c}{\mu_1}sech^2(\frac{1}{2}\sqrt{\frac{-c}{k^2\mu_3}}(ik(x-ct))). \tag{4.9}$$

When $\frac{c}{\mu_3} > 0$, we have the following periodic solutions:

$$U_3(x,t) = \frac{3c}{\mu_1}csc^2(\frac{1}{2}\sqrt{\frac{c}{k^2\mu_3}}(ik(x-ct))), \tag{4.10}$$

$$U_4(x,t) = \frac{3c}{\mu_1}sec^2(\frac{1}{2}\sqrt{\frac{c}{k^2\mu_3}}(ik(x-ct))). \tag{4.11}$$

4.2. **The Kudryashov method to solve the complex KdV equation.** The pole order of Eq. (4.3) is $N = 2$. So we look for solution of Eq. (4.3) in the following form

$$U(z) = b_0 + b_1 Q + b_2 Q^2 \tag{4.12}$$

Substituting Eq. (4.12) into Eq. (4.3), we obtain the system of algebraic equations in the following form

$$\begin{aligned}
&Q^0 : cb_0 - \frac{\mu_1}{2} b_0^2 = 0,\\
&Q^1 : cb_1 - \mu_1 b_0 b_1 + k^2 \mu_3 b_1 = 0,\\
&Q^2 : cb_2 - \frac{\mu_1}{2}(b_1^2 + 2b_0 b_2) + k^2 \mu_3(-3b_1 + 4b_2) = 0,\\
&Q^3 : -\mu_1 b_1 b_2 + k^2 \mu_3 (2b_1 - 10b_2) = 0,\\
&Q^4 : -\frac{\mu_1}{2} b_2^2 + 6k^2 \mu_3 b_2 = 0.
\end{aligned}$$

Solving the algebraic equations above, yields:

$$b_0 = \frac{2k^2\mu_3}{\mu_1},\ b_1 = -\frac{12k^2\mu_3}{\mu_1},\ b_2 = \frac{12k^2\mu_3}{\mu_1},\ c = k^2\mu_3. \tag{4.13}$$

Using ansatz given by Eq. (4.12), we obtain the following travelling wave solution of Eq. (4.3)

$$U(z) = \frac{2k^2\mu_3}{\mu_1} - \frac{12k^2\mu_3}{\mu_1}\left(\frac{1}{1 + dexp(z)}\right) + \frac{12k^2\mu_3}{\mu_1}\left(\frac{1}{1 + dexp(z)}^2\right). \tag{4.14}$$

where k is arbitrary constant.
Then the exact solution to Eq. (1.3) is written as

$$\begin{aligned}
U(x,t) = {} & \frac{2k^2\mu_3}{\mu_1} - \frac{12k^2\mu_3}{\mu_1}\left(\frac{1}{1 + dexp(ik(x - k^2\mu_3 t))}\right)\\
& + \frac{12k^2\mu_3}{\mu_1}\left(\frac{1}{1 + dexp(ik(x - k^2\mu_3 t))}^2\right).
\end{aligned}$$

5. Conclusions

Modification of truncated expansion method and the functional variable method are applied successfully for solving the complex KdV equation. Compared to the methods used before, one can see that these methods are direct, concise and effective. Moreover, the methods can also be used to many other nonlinear evolution equations.

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