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Spectral collocation method based on special functions for solving nonlinear high-order pantograph equations

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Abstract

In this paper, a spectral collocation method for solving nonlinear pantograph type delay differential equations is presented. The basis functions used for the spectral analysis are based on Chebyshev, Legendre, and Jacobi polynomials. By using the collocation points and operations matrices of required functions such as derivative functions and delays of unknown functions, the method transforms the problem into a system of nonlinear algebraic equations. The solutions of this nonlinear system determine the coefficients of the assumed solution. The method is explained by numerical examples and the results are compared with the available methods in the literature. It is seen from the applications that our method gives more efficient results than that of the reported methods.

Keywords. Nonlinear pantograph equations, Collocation method, Spectral method.

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1. Introduction

Delay differential equations (DDEs) and functional differential equations have an important place in science and engineering as many of the problems in these fields are characterized by such equations. Some of the examples include the models regarding the spread of diseases, subjects in climate science, applications regarding quantum dot lasers, and electronics engineering. The pantograph equation which is a class of first order delay differential equations is defined by

$$u'(x) = au(x) + bu(qx),\ x \in I = [0, T],\ 0 < q < 1. \tag{1.1}$$

with initial condition $u(0) = u_0$. The above equation is emerged as a mathematical modeling of the wave motion in the current line between an electric locomotive and an upper catheter wire. Later, this equation was developed to the following generalized delay differential form

$$y^{(m)}(x) + \sum_{j=0}^{J}\sum_{k=0}^{m-1} P_{jk}(x)y^{(k)}(\lambda_{jk}x + \mu_{jk}) = f(x),\ 0 \le x \le b, \tag{1.2}$$

and its solutions have been studied by various authors. The above generalized Pantograph equation has vast application in quantum calculus [6], population growth [17, 18], electric locomotive [20], control theory [26], dynamics of neural networks, etc.

Solutions to lower versions of these equations have been done using Berstein polynomial [11], First Boubaker polynomials [3], Gennochi polynomial [10] and orthoexponential functions collocation method [2]. In 2017, Rahimkhani studied the Pantograph equations using the Bernoulli wavelet method [22]. In 2018, Chang studied first order linear pantograph equation using the modified Chebyshev collocation method [28]. The Taylor operation method [30] was

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used by Yuzbasi to study the equation. In 2019, Gumgum studied different cases of neutral delay pantograph equations using the Legendre wavelets method [8] and Lucas collocation method of nonlinear delay differential equations [7]. Pantograph delay differential equations were solved using fractional-order hybrid Bessel functions by Dehestani et al [4] in 2020. Wang solved the nonlinear Pantograph equation using Jacobi spectral approximation [29] and shifted Chebyshev polynomials [27]. In 2021, Jafari et al. solved the delay differential equations using transfered Legendre polynomials. In the same year, Jaiswal and Yadav [12] proposed an algorithm using wavelets methods.

It is clear from the above survey that, any new numerical method that improves the earlier results with very good accuracy that can be obtained with less computational time and mathematical effort are always welcome in the literature. Hence, in the present study, we apply the spectral collocation method to study higher-order delay differential equations.

2. Present study

In this paper, we consider higher order linear/ nonlinear delay differential equations which can be represented in a generalized form as follows:

$$w^{(m)}(t) = \sum_{r=0}^{m_1}\sum_{s=0}^{m} p_{rs}(t)w^{(s)}(\alpha_{rs}t+\beta_{rs})) + \sum_{j=0}^{m_2}\sum_{k=0}^{m}\sum_{l=0}^{m} h_{jkl}(t)w^{(k)}(\alpha_{jkl}t+\beta_{jkl}))w^{(l)}(\gamma_{jkl}t+\mu_{jkl}) + f(t) \quad (2.1)$$

with initial conditions $w^{(\theta)} = V_i$, where $0 \leq \theta \leq m-1$ and $0 \leq i \leq m-1$
where $k \leq l$, $0 \leq m_1, m_2 \leq m$. $p_{rs}(t), h_{jkl}(t)$ can be either a constant or a variable function.

In this paper, we have studied higher and generalized pantograph equation given by (2.1). It is also noted that this type of nonlinear generalized pantograph equation has not yet been numerically studied by the spectral method using different polynomials in the literature. In our study, we have solved the higher-order delay differential equations using the spectral collocation method and obtained its solution using three different basis functions namely Chebyshev, Legendre, and Jacobi polynomials. In this paper, a set of twelve different cases of the general equation (2.1) has been considered to study for which analytical solutions exist. The twelve different cases are (i) First order linear delay differential equations with constant coefficients (ii) First order linear delay differential equations with variable coefficients (iii)First order nonlinear delay differential equations with variable coefficients (iv) Second order linear singularly delay equation with a variable coefficient (v) Second order nonlinear delay differential equations with constant coefficients (vi) Third order nonlinear delay differential equations with constant coefficients (vii) Third order linear delay differential equations with variable coefficients (viii) Third order nonlinear delay differential equations (ix) Third order linear time varying delay equation with constant coefficient (x) Fourth order linear delay differential equations with variable coefficients (xi) Fifth order linear delay differential equations with variable coefficients and (xii) Fifth order linear time varying delay differential equations with variable coefficients.

All these examples are compared with results available in the literature. We show that the spectral method is computationally efficient, easily implementable, and more accurate results can be obtained with least mathematical effort and the less computational time.

Our presentation is organized as follows: In the next section, the mathematical and numerical simplification of the governing equation are explained in a simple manner. In section 4, twelve examples of analytical solutions that are already available in the literature have been considered and solved using the spectral collocation method. The spectral results are compared with the analytical results and their error analysis are tabulated. Numerical results and discussions follow with several tables and figures. The paper ends with the conclusions highlighting the various findings from the present study.

3. Method of solution

Spectral collocation analysis is a very popular method of solution, especially for solving differential equations in the recent years [1, 9, 15, 16, 19, 24, 25]. The main idea behind this method is to write the spectral solution as the sum of truncated series using suitable basis functions. The set of basis functions we choose should be easy to compute and it should converge rapidly and the solution should have high accuracy when taking truncation N to be large. One of the main advantages of spectral methods is its fast rate of convergence, which is exponential for infinitely differentiable functions. The numerical scheme for generalized delay differential equation (2.1) with initial and boundary conditions is described in detail in this section:

The approximate solution for the interval [0, l] is given by

$$W_N(t) \approx W(t) \approx \sum_{n=0}^{N} a_n \phi_n((2t-l)/l). \tag{3.1}$$

Here, the number of collocation points are denoted by N and $\phi_n((2t-l)/l))$ gives the set of shifted basis functions. Let us denote $\phi_n((2t-l)/l))$ by $\phi_n(t)$ hereafter. As mentioned earlier, we have considered three different special functions such as Chebyshev polynomials $T_n((2t-l)/l))$, Legendre polynomials $L_n((2t-(b-a))/(b+a))$ and Jacobi polynomials $P_n^{(1,1)}((2t-l)/l))$ as basis functions.

The residual function is obtained on substituting approximate solution (3.1) in (2.1). Thus, the residual function is given by

$$\begin{aligned} Res(t) :=\ & \sum_{r=0}^{m_1}\sum_{s=0}^{m} p_{rs}(t)(\sum_{n=0}^{N} a_n \phi_n^{(s)}(\alpha_{rs}t+\beta_{rs}))+ \\ & \sum_{j=0}^{m_2}\sum_{k=0}^{m}\sum_{l=0}^{m} h_{jkl}(t)(\sum_{n=0}^{N} a_n\phi_n^{(k)}(\alpha_{jkl}t+\beta_{jkl}))(\sum_{n=0}^{N} a_n\phi_n^{(l)}(\gamma_{jkl}t+\mu_{jkl}))+f(t)-(\sum_{n=0}^{N} a_n\phi_n^{(m)}(t)); \end{aligned} \tag{3.2}$$

i.e.

$$Res(t) := \sum_{r=0}^{m_1}\sum_{s=0}^{m} p_{rs}(t)W^{(s)}(\alpha_{rs}t+\beta_{rs})+\sum_{j=0}^{m_2}\sum_{k=0}^{m-1}\sum_{l=0}^{m-1} h_{jkl}(t)W^{(k)}(\alpha_{jkl}t+\beta_{jkl})W^{(l)}(\gamma_{jkl}t+\mu_{jkl})+f(t)-W^{(m)}(t). \tag{3.3}$$

The essence of the present scheme is to force the residual function to be zero at certain sets of collocation points. These collocation points vary from one basis function to another. For the Chebyshev polynomials as basis functions, the set of collocation points considered are $\mathbf{t_i} = \mathbf{\frac{1}{2}(l-(l)cos(\pi * i)/N)}$, where l varies from 1 to $N-1$. For Legendre and Jacobi polynomials as basis functions, zeros of their first derivatives of the respective polynomials are chosen to be the collocation points. For the choice of basis functions and collocation points, the residual function (3.3) is converted as follows

$$\begin{aligned} Res(t_i) := & \sum_{r=0}^{m_1}\sum_{s=0}^{m} p_{rs}(t_i)W^{(s)}(\alpha_{rs}t_i+\beta_{rs})+\sum_{j=0}^{m_2}\sum_{k=0}^{m-1}\sum_{l=0}^{m-1} h_{jkl}(t_i)W^{(k)}(\alpha_{jkl}t_i+\beta_{jkl})W^{(l)}(\gamma_{jkl}t_i+\mu_{jkl}) \\ & + f(t_i) - W^{(m)}(t_i) \\ = & \ 0, \end{aligned} \tag{3.4}$$

where i varies from 0 to N. The simplified matrix form of (3.4) is given by:

$$\begin{aligned} \mathbf{R(t)} := & \sum_{r=0}^{m_1}\sum_{s=0}^{m} P_{rs}D^{(m)}(\alpha_{rs},\beta_{rs})A+\sum_{j=0}^{m_2}\sum_{k=0}^{m}\sum_{l=0}^{m}(H_{jkl}D^{(k)}(\alpha_{jkl},\beta_{jkl})A)D^{(l)}(\gamma_{jkl},\mu_{jkl})A \\ & + F - D^{(m)}A \\ = & \ [0], \end{aligned} \tag{3.5}$$

where

$$\mathbf{D^{(m)}} = \begin{bmatrix} \phi_0^{(m)}(t_0) & \phi_1^{(m)}(t_0) & \dots & \phi_N^{(m)}(t_0) \\ \vdots & \vdots & \dots & \vdots \\ \phi_0^{(m)}(t_N) & \phi_1^{(m)}(t_N) & \dots & \phi_N^{(m)}(t_N) \end{bmatrix}_{N+1\times N+1}, \quad \mathbf{P_{rs}} = \begin{bmatrix} p_{rs}(t_0) & 0 & 0 & \dots & 0 \\ 0 & p_{rs}(t_1) & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & p_{rs}(t_N) \end{bmatrix}_{N+1\times N+1},$$

$$\mathbf{H_{jkl}} = \begin{bmatrix} h_{jkl}(t_0) & 0 & 0 & \dots & 0 \\ 0 & h_{jkl}(t_1) & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & h_{jkl}(t_N) \end{bmatrix}_{N+1\times N+1}, \quad \mathbf{F} = \begin{bmatrix} f(t_0) & 0 & 0 & \dots & 0 \\ 0 & f(t_1) & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & f(t_N) \end{bmatrix}_{N+1\times N+1},$$

$$\mathbf{A} = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_N \end{bmatrix}_{N+1\times 1}, \quad \mathbf{R(t)} = \begin{bmatrix} Res(t_0) \\ Res(t_1) \\ \vdots \\ Res(t_N) \end{bmatrix}_{N+1\times 1},$$

where $\mathbf{D^{(m)}}$ is a differentiation matrix in which basis function are differentiated m times. $\mathbf{H_{jkl}}$,$\mathbf{P_{rs}}$ represents the coefficient matrix. $\mathbf{F}$ represents the non homogeneous function matrix. [0] is the $N+1\times 1$ matrix with zero entries.

$$\mathbf{D^{(m)}}(\alpha_{\mathbf{rs}}, \beta_{\mathbf{rs}}) = \begin{bmatrix} \phi_0^{(m)}(\alpha_{rs}t_0+\beta_{rs}) & \phi_1^{(m)}(\alpha_{rs}t_0+\beta_{rs}) & \dots & \phi_N^{(m)}(\alpha_{rs}t_0+\beta_{rs}) \\ \vdots & \vdots & \dots & \vdots \\ \phi_0^{(m)}(\alpha_{rs}t_N+\beta_{rs}) & \phi_1^{(m)}(\alpha_{rs}t_N+\beta_{rs}) & \dots & \phi_N^{(m)}(\alpha_{rs}t_N+\beta_{rs}) \end{bmatrix}_{N+1\times N+1},$$

$$\mathbf{D^{(k)}}(\alpha_{\mathbf{jkl}}, \beta_{\mathbf{jkl}}) = \begin{bmatrix} \phi_0^{(k)}(\alpha_{jkl}t_0+\beta_{jkl}) & \phi_1^{(k)}(\alpha_{jkl}t_0+\beta_{jkl}) & \dots & \phi_N^{(k)}(\alpha_{jkl}t_0+\beta_{jkl}) \\ \vdots & \vdots & \dots & \vdots \\ \phi_0^{(k)}(\alpha_{jkl}t_N+\beta_{jkl}) & \phi_1^{(k)}(\alpha_{jkl}t_N+\beta_{jkl}) & \dots & \phi_N^{(k)}(\alpha_{jkl}t_N+\beta_{jkl}) \end{bmatrix}_{N+1\times N+1},$$

$$\mathbf{D^{(l)}}(\gamma_{\mathbf{jkl}}, \mu_{\mathbf{jkl}}) = \begin{bmatrix} \phi_0^{(l)}(\gamma_{jkl}t_0+\mu_{jkl}) & \phi_1^{(l)}(\gamma_{jkl}t_0+\mu_{jkl}) & \dots & \phi_N^{(l)}(\gamma_{jkl}t_0+\mu_{jkl}) \\ \vdots & \vdots & \dots & \vdots \\ \phi_0^{(l)}(\gamma_{jkl}t_N+\mu_{jkl}) & \phi_1^{(l)}(\gamma_{jkl}t_N+\mu_{jkl}) & \dots & \phi_N^{(l)}(\gamma_{jkl}t_N+\mu_{jkl}) \end{bmatrix}_{N+1\times N+1},$$

where $\mathbf{D^{(m)}}(\alpha_{\mathbf{rs}}, \beta_{\mathbf{rs}})$, $\mathbf{D^{(k)}}(\alpha_{\mathbf{jkl}}, \beta_{\mathbf{jkl}})$, $\mathbf{D^{(l)}}(\gamma_{\mathbf{jkl}}, \mu_{\mathbf{jkl}})$ are delayed differentiation matrix. Thus the more simplified form of (3.5) is given by (3.6) as follows

$$\mathbf{R(t)} := \sum_{\mathbf{r=0}}^{\mathbf{m_1}} \sum_{\mathbf{s=0}}^{\mathbf{m}} \mathbf{P_{rs}} \mathbf{Y^{(m)}}(\alpha_{\mathbf{rs}}, \beta_{\mathbf{rs}}) + \sum_{\mathbf{j=0}}^{\mathbf{m_2}} \sum_{\mathbf{k=0}}^{\mathbf{m-1}} \sum_{\mathbf{l=0}}^{\mathbf{m-1}} \mathbf{H_{jkl}} \mathbf{G_{jkl}^{(k,l)}} + \mathbf{F} - \mathbf{Y^{(m)}} = [0]. \tag{3.6}$$

where

$$\mathbf{Y^{(m)}} = \mathbf{D^{(m)}A},\ \mathbf{Y^{(m)}}(\alpha_{\mathbf{rs}}, \beta_{\mathbf{rs}}) = \mathbf{D^{(m)}}(\alpha_{\mathbf{rs}}, \beta_{\mathbf{rs}})\mathbf{A},\ \mathbf{Y^{(k)}}(\alpha_{\mathbf{jkl}}, \beta_{\mathbf{jkl}}) = \mathbf{D^{(k)}}(\alpha_{\mathbf{jkl}}, \beta_{\mathbf{jkl}})\mathbf{A}, \tag{3.7}$$

$$\mathbf{Y^{(l)}}(\gamma_{\mathbf{jkl}}, \mu_{\mathbf{jkl}}) = \mathbf{D^{(l)}}(\gamma_{\mathbf{jkl}}, \mu_{\mathbf{jkl}})\mathbf{A},\ \mathbf{G^{(k,l)}} = \mathbf{H^{(k)}}(\alpha_{\mathbf{jkl}}, \beta_{\mathbf{jkl}}) \times \mathbf{Y^{(l)}}(\gamma_{\mathbf{jkl}}, \mu_{\mathbf{jkl}}), \tag{3.8}$$

where

$$\mathbf{H^{(k)}}(\alpha_{\mathbf{jkl}}, \beta_{\mathbf{jkl}}) = \begin{bmatrix} (\mathbf{Y^{(k)}}(\alpha_{\mathbf{jkl}}, \beta_{\mathbf{jkl}}))_{\mathbf{1}} & 0 & \dots & 0 \\ 0 & (\mathbf{Y^{(k)}}(\alpha_{\mathbf{jkl}}, \beta_{\mathbf{jkl}}))_{\mathbf{2}} & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & (\mathbf{Y^{(k)}}(\alpha_{\mathbf{jkl}}, \beta_{\mathbf{jkl}}))_{\mathbf{N+1}} \end{bmatrix}_{N+1\times N+1}$$

where $(\mathbf{Y^{(k)}}(\alpha_{\mathbf{jkl}}, \beta_{\mathbf{jkl}}))_{\mathbf{i}}$ represents the i^{th} vector of the matrix $\mathbf{Y^{(k)}}(\alpha_{\mathbf{jkl}}, \beta_{\mathbf{jkl}})$.

Thus, on substituting the spectral solution (3.1) in the left hand side of the initial conditions, we have

$$I_0 := \sum_{n=0}^{N} a_n \phi_n(t_0); \qquad I_1 := \sum_{n=0}^{N} a_n \phi_n^{(1)}(t_0); \qquad \dots; \qquad I_{m-1} := \sum_{n=0}^{N} a_n \phi_n^{(m-1)}(t_0). \tag{3.9}$$

These initial conditions are written in matrix form as follows: $\mathbf{Int} := \overline{\mathbf{D}}\mathbf{A}$ and $\overline{\mathbf{D}}\mathbf{A} = \mathbf{V}$.
Thus, we have $\mathbf{Int} = \mathbf{V}$. where

$$\overline{\mathbf{D}} = \begin{bmatrix} \phi_0(t_0) & \phi_1(t_0) & \dots & \phi_N(t_0) \\ \vdots & \vdots & \dots & \vdots \\ \phi_0^{(m-1)}(t_N) & \phi_1^{(m-1)}(t_N) & \dots & \phi_N^{(m-1)}(t_N) \end{bmatrix}_{m\times N+1}, \mathbf{Int} = \begin{bmatrix} I_0 \\ I_1 \\ \vdots \\ I_{m-1} \end{bmatrix}_{m\times 1}, \mathbf{V} = \begin{bmatrix} V_0 \\ V_1 \\ \vdots \\ V_{m-1} \end{bmatrix}_{m\times 1}.$$

Since, we have m initial conditions. The last m rows of residual matrix (3.6) are replaced by the m initial conditions. On total, now we have matrix of the form

$$\begin{bmatrix} Res(t_0) \\ Res(t_1) \\ \vdots \\ Res(t_{n-m}) \\ I_0 \\ \vdots \\ I_{m-1} \end{bmatrix}_{N+1\times 1} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ V_0 \\ \vdots \\ V_{m-1} \end{bmatrix}_{N+1\times 1}.$$

Thus, we get $N+1$ algebraic equations with $N+1$ unknowns. The present method converts the higher order delayed differential equation into algebraic equations which is solved using Newton's method. The unknown coefficients a_n, where i varies from 0 to N are obtained on solving these linear/nonlinear algebraic equations. On substituting these unknown coefficients in equations (3.1), yields spectral solution for $W_N(t)$.

4. Numerical Applications

In this section, we have considered twelve different examples for which analytical solution already exists. The initial conditions are extracted from the analytical solutions. Here, the solutions are calculated over the domain $[0, 1]$. All the computations are done using Maple 18. The absolute and residual error play a vital role in determining the accuracy of the method. The absolute error is given by

- Absolute Error $\mathbf{E_N(t_j)} := |\mathbf{W_N(t_j)} - W(t_j)|$,
- Maximum Absolute Error $\mathbf{MAE_N} := \max_{0\le t_j\le 1} |\mathbf{E_N(t_j)}|$,
- $\mathbf{L_2}$ **norm** $:= \int_0^1 \mathbf{W_N(t)}dt$.

where $\mathbf{W_N(t_j)}$ represents the spectral solutions with 'N' collocation points for a given t_j and $W(t_j)$ is the exact solution obtained from the literature for a given t_j.

Example 4.1. *Let us consider* (2.1) *with* $m = 1$, $m_1 = 1$, $p_{00} = -1$, $\alpha_{00} = 0.8$, $p_{10} = 1$, $\alpha_{10} = 1$ *which is a linear delay differential equation with constant coefficients and proportional delay given by [13, 28]:*

$$w^{(1)}(t) = -w(0.8t) + w(t), \tag{4.1}$$

with the initial condition $w(0) = 1$.

This model arise in the mathematical modeling of the wave motion in the supply line to an overhead current collector (pantograph) of an electric locomotive. As an illustration, let us consider for $N = 6$ with Chebyshev polynomials as basis function. The residual function for (4.1) is given by

$$\mathbf{R(t)} := \mathbf{P_{00}D(0.8t, 0)A} + \mathbf{P_{01}DA} - \mathbf{D^{(1)}A}. \tag{4.2}$$

Using the set of collocation points $t_0 = 0, t_1 = 0.066987, t_2 = 0.25, t_3 = 0.5, t_4 = 0.75, t_5 = 0.93301, t_6 = 1$, the first order and second order derivative matrices are derived as follows:

$$\mathbf{D} = \begin{bmatrix} 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ 1 & -0.866026 & 0.500002 & -0.000035773 & -0.499996 & 0.866022 & -1. \\ 1 & -0.5 & -0.5 & 1. & -0.5 & -0.5 & 1. \\ 1 & 0. & -1. & 0. & 1. & 0. & -1. \\ 1 & 0.5 & -0.5 & -1. & -0.5 & 0.5 & 1. \\ 1 & 0.86602 & 0.499981 & -0.0000324224 & -0.500037 & -0.866052 & -1. \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix},$$

$$\mathbf{D^{(1)}} = \begin{bmatrix} 0 & 2 & -8 & 18 & -32 & 50 & -72 \\ 0 & 2 & -6.92821 & 12. & -13.8565 & 10.0001 & -0.000171711 \\ 0 & 2 & -4. & 0. & 8. & -10. & 0. \\ 0 & 2 & 0. & -6. & 0. & 10. & 0. \\ 0 & 2 & 4. & 0. & -8. & -10. & 0. \\ 0 & 2 & 6.92816 & 11.9998 & 13.8558 & 9.99888 & -0.00155625 \\ 0 & 2 & 8 & 18 & 32 & 50 & 72 \end{bmatrix}_{7\times 7},$$

$$\mathbf{P_{00}} = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}_{7\times 7}, \quad \mathbf{P_{10}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}_{7\times 7}, \quad \mathbf{A} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix}_{7\times 1},$$

$$\mathbf{D(0.8t, 0)} = \begin{bmatrix} 1. & -1. & 1. & -1. & 1. & -1. & 1. \\ 1. & -0.892821 & 0.594258 & -0.168311 & -0.293715 & 0.692781 & -0.943343 \\ 1. & -0.6 & -0.28 & 0.936 & -0.8432 & 0.07584 & 0.752192 \\ 1. & -0.2 & -0.92 & 0.568 & 0.6928 & -0.84512 & -0.354752 \\ 1. & 0.2 & -0.92 & -0.568 & 0.6928 & 0.84512 & -0.354752 \\ 1. & 0.492816 & -0.514265 & -0.999692 & -0.471063 & 0.535397 & 0.998767 \\ 1. & 0.6 & -0.28 & -0.936 & -0.8432 & -0.07584 & 0.752192 \end{bmatrix}_{7\times 7},$$

$$\mathbf{R(t)} = \begin{bmatrix} Res(t_0) \\ Res(t_1) \\ Res(t_2) \\ Res(t_3) \\ Res(t_4) \\ Res(t_5) \\ Res(t_6) \end{bmatrix}_{7\times 1}.$$

The last row of residual error matrix $\mathbf{R(t)}$ is replaced by the initial condition I_0 given by

$$I_0 := \overline{\mathbf{D}}\mathbf{A} = \mathbf{1}, \tag{4.3}$$

TABLE 1. Comparison of spectral solution with SOB-Operational[13], MCCM[28], TLPM [14] and OMCP[21] for Example 4.1.

t	SOB-Operational [13] $N=6$	MCCM [28] $N=6$	TLPM [14] $N=6$	Present Method $N=6$	OMCP [21] $N=15$	Present Method $N=15$
0.2	0.66469078	0.66469101	0.6646906619	0.664691281	0.66469100082	0.66469100083
0.4	0.43356098	0.43356077	0.4335614390	0.433560399	0.43356077877	0.43356077878
0.6	0.27648212	0.27648233	0.2764814803	0.276482249	0.27648233022	0 .27648233022
0.8	0.17148433	0.17148412	0.1714843782	0.171484942	0.17148411197	0.17148411198
1	0.10267077	0.10267013	0.1026700543	0.10267150	0.10267012657	0.10267012657
CPU Time			0.047		0.077	

TABLE 2. Comparison of CPU Time used for Present method with SOB-Operational [13] and TM [30] for Example 4.1.

Method	CPU Time (seconds)	
	N = 5	N = 8
Present method	0.046	0.048
SOB-Operational[13]	0.8892	0.9516
Taylor Method [30]	0.2341	0.2896

where

$$\overline{\mathbf{D}} = \begin{bmatrix} 1. & -1. & 1. & -1. & 1. & -1. & 1. \end{bmatrix}.$$

Thus, on combining residual functions with initial conditions, we have 7 algebraic equations with 7 unknowns variables. On solving these equations, we get the final spectral solution

$$\begin{aligned} \mathbf{W_N}(t) = {} & 0.01389269182017972142 1t^6 - 0.09580064095474856970 2t^5 + 0.36780363537832919158t^4 \\ & - 0.98317281543314782037t^3 + 1.79994862916711517 01t^2 - 2.0t + 0.99999999999999999997. \end{aligned} \tag{4.4}$$

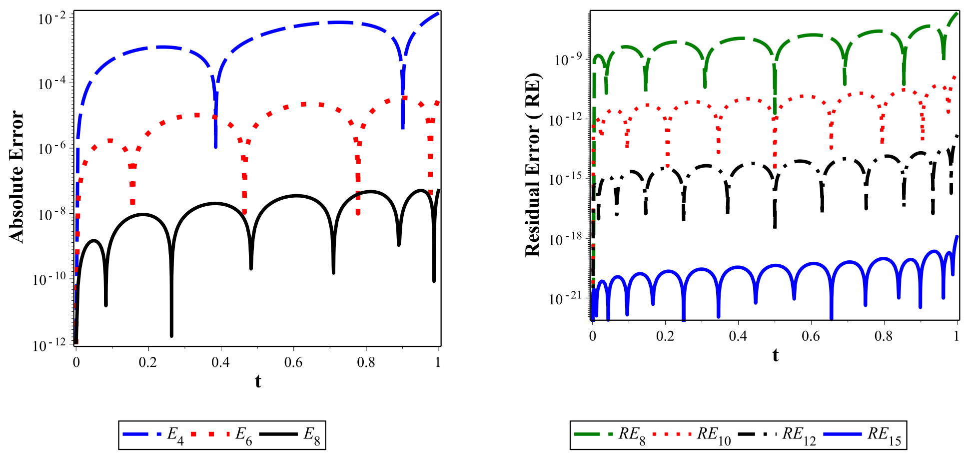
Table 1 compares the spectral solution with other numerical methods such as Modified Chebyshev Collocation Method (MCCM)[28], numerical method using shifted orthonormal Bernstein polynomials (SOB-Operational)[13], Transfered Legendere Pseduospectral method [TLPM] [14] and Operational Method using Chebyshev Polynomial (OMCP)[21] for different values of N. The CPU time used for different values of N is compared with other methods such as SOB-Operational [13] and Taylor Method (TM)[30] is tabulated in Table 2. The computational time in the Tables 1 and 2 evidently highlights that the present method is computationally efficient. The absolute and residual error plot for different vales of N is depicted in Figure 1. Figure 1(a) shows that as N increases from 4 to 8, the absolute error decreases from $O(10^{-2})$ to $O(10^{-8})$. In Figure 1(b) , the residual error elevates from $O(10^{-8})$ to $O(10^{-18})$ as N raises from 8 to 15. This reveals that as N increase, there is decrease in the error values. (i.e.) the error converges.

Example 4.2. *Let us consider* (2.1) *with* $m = 1$*,* $m_1 = 1$*,* $p_{00} = 0.5e^{0.5t}$ *,* $\alpha_{00} = 0.5$*,* $p_{10} = 0.5$*,* $\alpha_{10} = 1$ *which is a linear delay equation with variable coefficient given by [21, 28]:*

$$w^{(1)}(t) = 0.5e^{0.5t}w(0.5t) + 0.5w(t), \tag{4.5}$$

with the initial condition $w(0) = 1$ *and the exact solution* $w(t) = e^t$*.*

Table 3 compares the absolute error obtained using present method with MCCM [28], OMCP [21] for $N = 8$ and with Shifted Chebyshev Polynomial (SCP) [27] for $N = 11$. It can be noted from the table that at same number of collocation points, the present method gives good accuracy compared to other methods in the literature. Table 4 represents the L_2 norm error for different values of N with respect to all the three different basis function. As N increases, the convergence of the error can be seen in these tables. Figure 2(a) depicts the residual error for different values of N and Figure 2(b) compares the absolute error obtained using present method with that of MCCM [28] and Operational Matrix method using Genocchi polynomials (OGPM)[10] for $N = 16$. It can be seen that for $N = 16$, absolute error for MCCM, OGPM is of order 10^{-15} and 10^{-19} respectively. But for the same N, present method shows absolute error of order 10^{-24}. This shows the novelty of the present method.



(a) Absolute error plot. (b) Residual error plot.

FIGURE 1. Absolute and Residual error plot for different values of N for Example 4.1.

TABLE 3. Comparison of absolute error with MCCM [28], OMCP [21] and SCP [27] for Example 4.2.

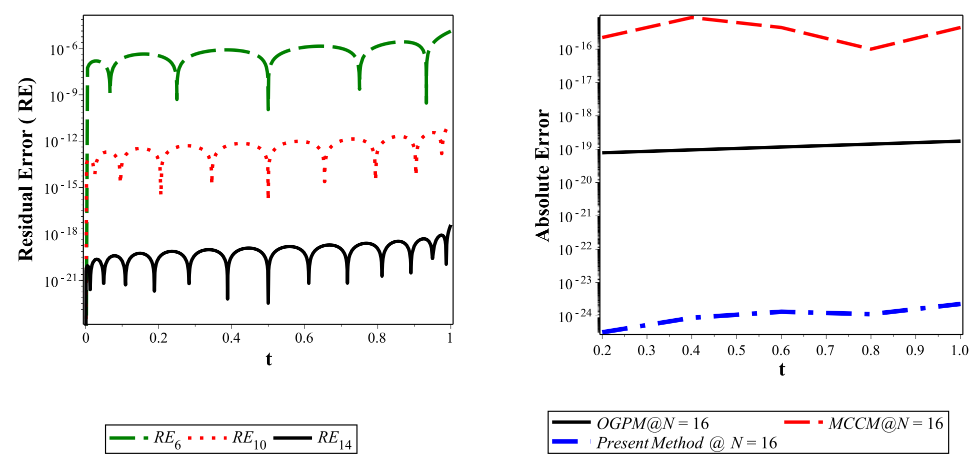
t	MCCM[28]	OMCP [21]	Present Method at N = 8			SCP [27]	Present method
	$N = 8$	$N = 8$	Chebyshev	Legendre	Jacobi	$N = 11$	$N = 11$
0.2	$7.53 * 10^{-11}$	$0.314 * 10^{-7}$	$7.35341 * 10^{-12}$	$1.17824 * 10^{-11}$	$7.88253 * 10^{-13}$	$1.80 * 10^{-14}$	$1.876 * 10^{-16}$
0.4	$1.20 * 10^{-9}$	$0.109 * 10^{-6}$	$8.83223 * 10^{-12}$	$2.04758 * 10^{-13}$	$2.07973 * 10^{-11}$	$2.09 * 10^{-14}$	$5.914 * 10^{-16}$
0.6	$1.39 * 10^{-9}$	$0.818 * 10^{-7}$	$9.09392 * 10^{-12}$	$1.35129 * 10^{-11}$	$3.22923 * 10^{-11}$	$2.42 * 10^{-14}$	$8.276 * 10^{-16}$
0.8	$1.39 * 10^{-10}$	$0.506 * 10^{-7}$	$5.301778 * 10^{-11}$	$6.42538 * 10^{-11}$	$3.54560 * 10^{-11}$	$2.93 * 10^{-14}$	$1.064 * 10^{-15}$
1	$7.99 * 10^{-10}$	$0.123 * 10^{-6}$	$1.587427 * 10^{-10}$	$3.89256 * 10^{-10}$	$9.91012 * 10^{-10}$	$3.20 * 10^{-14}$	$1.813 * 10^{-15}$

TABLE 4. L_2 norm error for different values of N for Example 4.2.

	Present Method		
N	Chebyshev	Legendre	Jacobi
4	$5.031 * 10^{-3}$	$4.843 * 10^{-3}$	$6.471 * 10^{-3}$
8	$6.159 * 10^{-9}$	$5.745 * 10^{-9}$	$1.03768 * 10^{-8}$
12	$1.3632 * 10^{-15}$	$1.2756 * 10^{-15}$	$2.753 * 10^{-15}$
16	$2.9130 * 10^{-23}$	$8.620 * 10^{-23}$	$2.1209 * 10^{-22}$

Table 5. Comparison of absolute error with FBPCM [3] for Example 4.3.

t	FBPCM [3]	Present Method at N = 9		
	$N = 9$	Chebyshev	Legendre	Jacobi
0.01	$2.05 * 10^{-11}$	$1.81105 * 10^{-12}$	$1.13350 * 10^{-12}$	$2.10893 * 10^{-12}$
0.04	$9.83 * 10^{-12}$	$7.03202 * 10^{-13}$	$5.72494 * 10^{-12}$	$1.54834 * 10^{-11}$
0.09	$4.45 * 10^{-11}$	$1.13635 * 10^{-11}$	$1.13350 * 10^{-12}$	$1.34538 * 10^{-11}$
0.16	$1.41 * 10^{-11}$	$5.62434 * 10^{-12}$	$5.97957 * 10^{-12}$	$1.65052 * 10^{-12}$
0.25	$6.54 * 10^{-11}$	$3.12437 * 10^{-11}$	$1.00289 * 10^{-11}$	$2.29982 * 10^{-11}$
0.36	$1.19 * 10^{-10}$	$2.97628 * 10^{-11}$	$2.42276 * 10^{-11}$	$8.01747 * 10^{-12}$
0.49	$1.61 * 10^{-10}$	$1.08094 * 10^{-11}$	$1.59384 * 10^{-11}$	$9.12145 * 10^{-12}$
0.64	$3.37 * 10^{-10}$	$3.483548 * 10^{-11}$	$6.60569 * 10^{-12}$	$1.35250 * 10^{-11}$
0.81	$3.81 * 10^{-10}$	$2.39195 * 10^{-11}$	$1.78993 * 10^{-11}$	$4.36959 * 10^{-11}$
CPU Time	0.789			



(a) Residual error plot for different values of N.

(b) Comparison of absolute error with MCCM and OGPM.

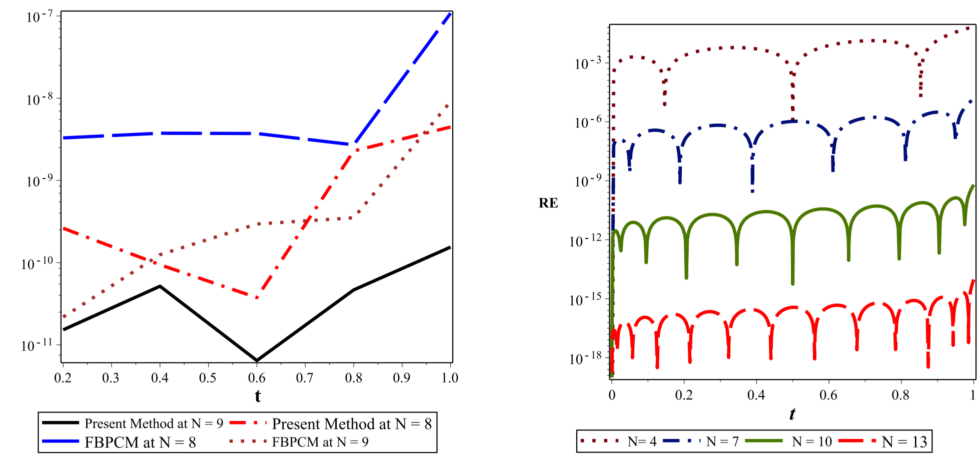
Figure 2. Residual and absolute error plot for Example 4.2.

Example 4.3. *Let us consider* (2.1) *with* $m = 1$, $m_1 = 0$, $m_2 = 0$, $p_{00} = 1$, $\alpha_{00} = 1$, $h_{000} = t$, $\alpha_{000} = \gamma_{000} = 0.5$ *which is a nonlinear delay equation with variable coefficient given by [3] :*

$$w^{(1)}(t) = w(t) + tw(0.5t)^2 + f(t), \tag{4.6}$$

with the initial condition $w(0) = -1$ *and the exact solution* $9te^{t-1} - 1$. *The f(t) is given by* $1 + 9e^{(} - 1 + t) - t(-1 + 4.5e^{(} - 1 + 0.5t)t)2$.

Table 5 compares the absolute error obtained using present method with First Boubaker Polynomials Collocation Method (FBPCM) [3] for different collocation points t_i . Figure 3(a) displays the comparative plot of absolute errors for $N = 8$ and 9 obtained using present method with FBPCM . Figure 3(b) depicts the residual error plot for N using present method. In the plots, we observe that for same number of collocation points, the present method shows maximum absolute error of order 10^{-10}, whereas FBPCM shows accuracy of 10^{-8}. Thus, Obtaining good accuracy at less computational time compared to the other methods enhances the richness of the present method.



(a) Comparison of absolute error plot with FBPCM.

(b) Residual error plot for different values of N.

FIGURE 3. Absolute and Residual error plot for Example 4.3.

TABLE 6. Absolute error for Example 4.4.

t	Present Method at $N = 8$		
	Chebyshev	Legendre	Jacobi
0.2	$1.1392304 * 10^{-8}$	$1.5055636 * 10^{-9}$	$8.3082386 * 10^{-10}$
0.4	$5.8505494 * 10^{-8}$	$7.4097943 * 10^{-9}$	$3.8623418 * 10^{-9}$
0.6	$7.5131327 * 10^{-8}$	$9.6763074 * 10^{-9}$	$5.0340879 * 10^{-9}$
0.8	$5.45657523 * 10^{-8}$	$6.9519739 * 10^{-9}$	$3.572930 * 10^{-9}$

Example 4.4. *Let us consider* (2.1) *with* $m = 2$, $m_1 = 1$, $p_{00} = \frac{1}{t-1}$, $p_{01} = \frac{1}{t}$, $p_{11} = \frac{1}{t^2}$, $\alpha_{00} = 1$, $\alpha_{01} = 0.5$, $\alpha_{11} = 0.25$ *which is a second order linear singularly delay equation with variable coefficient is given by:*

$$w^{(2)}(t) = \frac{1}{t}w^{(1)}(\frac{t}{2}) + \frac{1}{t^2}w^{(1)}(\frac{t}{4}) + \frac{1}{t-1}w(t) - f(t), \tag{4.7}$$

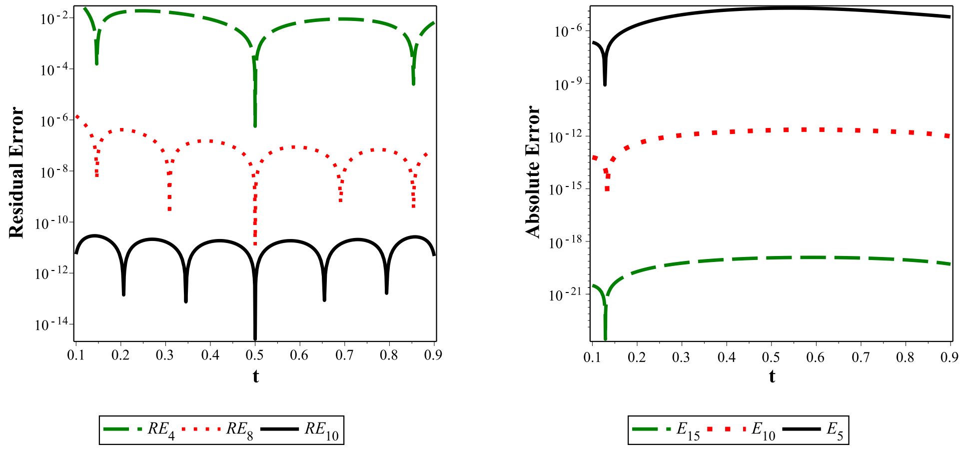
with the initial conditions $w(0) = 1$, $w^{(1)}(0) = 1$. *The exact solution of the problem is given by* e^t . *The* $f(t)$ *is obtained on substituting the exact solution in the given problem.*

Table 6 displays the absolute error for $N = 8$ for three different basis functions. Figure 4 depicts the residual and absolute error plot for different value of N. Figure 4(a) shows that the error is of order 10^{-2} for $N = 4$, which later declines to $O(10^{-21})$ when N raises to 10. Similarly in Figure 4(b), the change in residual error from $O(10^{-6})$ to $O(10^{-21})$ as N increases from 5 to 15 is noted. This change shows the occurrence of convergence of error as N increases.

Example 4.5. *Let us consider* (2.1) *with* $m = 2$, $m_2 = 0$, $h_{000} = -2$, $f(t) = 1$, $\alpha_{000} = \gamma_{000} = 0.5$ *which is a second order nonlinear delay equation given by [22]:*

$$w^{(2)}(t) = 1 - 2w^2(\frac{t}{2}), \tag{4.8}$$

with initial conditions $w(0) = 1$ *and* $w^{(1)}(0) = 0$. *The exact solution is given by* $w(t) = cos(t)$.



(a) Residual error plot. (b) Absolute error plot.

FIGURE 4. Absolute and residual error plot for Example 4.4.

TABLE 7. Comparison of absolute error with OMBWM [23] and JSA for Example 4.5.

t	OMBWM [23]	Present Method at N = 8			JSA [29]	Present method
	$k = 2, M = 7$	Chebyshev	Legendre	Jacobi	$N = 10$	$N = 10$
0.2	$1.05 * 10^{-10}$	$3.490365 * 10^{-12}$	$2.44564 * 10^{-12}$	$2.10893 * 10^{-12}$	$6.62 * 10^{-13}$	$6.2868 * 10^{-16}$
0.4	$3.21 * 10^{-11}$	$8.93559 * 10^{-12}$	$7.03235 * 10^{-12}$	$1.54834 * 10^{-11}$	$6.93 * 10^{-12}$	$6.9251 * 10^{-15}$
0.6	$3.81 * 10^{-11}$	$1.13635 * 10^{-11}$	$1.45763 * 10^{-11}$	$1.34538 * 10^{-11}$	$2.89 * 10^{-12}$	$9.8200 * 10^{-15}$
0.8	$1.31 * 10^{-6}$	$5.62434 * 10^{-12}$	$1.06428 * 10^{-11}$	$1.65052 * 10^{-12}$	$3.95 * 10^{-11}$	$1.9722 * 10^{-14}$
1	$1.82 * 10^{-6}$	$3.12437 * 10^{-11}$	$1.97986 * 10^{-9}$	$2.29982 * 10^{-11}$	-	$1.0758 * 10^{-12}$

Table 7 represents the comparison of absolute error for $K = 2$, $M = 7$ with Operational matrix based on Bernoulli Wavelets Method (OMBWM) [23] and $N = 10$ with Jacobi Spectral Approximation Method (JSA) [29]. Both the comparison shows that the present method with considered basis functions shows good accuracy compared to Bernoulli basis functions.

Example 4.6. *Let us consider* (2.1) *with* $m = 3$, $m_2 = 0$, $h_{000} = 2$, $f(t) = -1$, $\alpha_{000} = \gamma_{000} = 0.5$ *which is a third order nonlinear pantograph equation with constant coefficient given by [10]:*

$$w^{(3)}(t) = 2w^2(\frac{t}{2}) - 1 \tag{4.9}$$

with initial conditions $w(0) = 0$, $w^{(1)}(0) = 1$, $w^(2)(0) = -1$. *The exact solution is given by* $w(t) = sin(t)$.

Table 8 compares the absolute error obtained using present method with Fractional-Order Hybrid Bessel Functions (FHBFs) [4] for $N = 8$ and OGPM [10] for $N = 15$. For same number of collocation points, the present method shows maximum absolute error of order 10^{-20} whereas OGPM shows of order 10^{-16}. This table reveals that present method shows good resolution of results compared to [10].

Example 4.7. *Let us consider* (2.1) *with* $m = 3$, $m_1 = 0$, $p_{00} = -1$, $\alpha_{00} = 0.5$, $p_{01} = -1$,$\alpha_{01} = 1$, $p_{02} = t$, $\alpha_{02} = 2$, $f(t) = tcos(2t) + cos(0.5t)$ *which is a third order linear delay equation with variable coefficient given by [2, 11]:*

$$w^{(3)}(t) = tw^{(2)}(2t) - w^{(1)}(t) - w(0.5t) + tcos(2t) + cos(0.5t) \tag{4.10}$$

TABLE 8. Comparison of absolute error with FHBF [4] and OGPM [10] for Example 4.6.

t	FHBF [4]	Present Method	OGPM [10]	Present Method at $N = 15$		
	$N = 8$	$N = 8$	$N = 15$	Chebyshev	Legendre	Jacobi
0.2	$7.97 * 10^{-8}$	$5.171 * 10^{-12}$	$3.22 * 10^{-18}$	$1.22142 * 10^{-24}$	$1.68347 * 10^{-25}$	$9.07571 * 10^{-24}$
0.4	$4.90 * 10^{-8}$	$1.707 * 10^{-11}$	$7.19 * 10^{-18}$	$7.68206 * 10^{-24}$	$3.62382 * 10^{-24}$	$3.34660 * 10^{-23}$
0.6	$1.52 * 10^{-7}$	$2.881 * 10^{-11}$	$1.42 * 10^{-16}$	$1.90367 * 10^{-23}$	$1.27697 * 10^{-23}$	$6.98930 * 10^{-23}$
0.8	$5.09 * 10^{-7}$	$3.087 * 10^{-10}$	$3.78 * 10^{-17}$	$1.11994 * 10^{-22}$	$9.48676 * 10^{-23}$	$8.44557 * 10^{-23}$
1	$9.78 * 10^{-7}$	$3.376 * 10^{-8}$	$7.97 * 10^{-8}$	$5.99791 * 10^{-20}$	$8.25844 * 10^{-20}$	$1.34563 * 10^{-19}$

TABLE 9. Comparison of spectral solution with OECM [2], Berstein series [11] and Exact for Example 4.7.

t	OECM [2]	Berstein series [11]	Present Method	Exact
	N = 5	N = 6	N = 5	
0.2	0.979528737	.9800663232	0.9800666	0.9800666
0.4	0.9180374858	0.9210523648	0.9210611	0.9210610
0.6	0.8183205412	0.8252705728	0.8253474	0.8253356
0.8	0.6871599035	0.6964380672	0.6968226	0.6967067
1	0.5381733576	0.5395000000	0.5408811	0.5403023

TABLE 10. Absolute error at $N = 8$ for Example 4.8.

t	Present Method		
	Chebyshev	Legendre	Jacobi
0.2	$1.4313681 * 10^{-12}$	$4.1173206 * 10^{-13}$	$3.19232111 * 10^{-12}$
0.4	$5.4352760 * 10^{-12}$	$4.2430693 * 10^{-12}$	$1.46283282 * 10^{-11}$
0.6	$2.8166072 * 10^{-11}$	$1.6375125 * 10^{-11}$	$2.48430479 * 10^{-11}$
0.8	$1.0789954 * 10^{-10}$	$6.8488563 * 10^{-11}$	$5.5572141 * 10^{-11}$
1	$1.2434409 * 10^{-8}$	$1.31981840 * 10^{-8}$	$1.4059401 * 10^{-8}$

The initial conditions are given by $w(0) = 1$, $w^{(1)}(0) = 0$,$w^{(2)}(0) = -1$. *The exact solution is given by* $w(t) = cos(t)$.

Comparison of spectral solution with Exact, Berstein series [11] and Orthogonal Exponential Collocation Method (OECM) [2] for Example 4.7 is tabulated in Table 9. It can be noted from table that the present method captures accuracy upto 3 digits for N = 5 whereas OECM captures only upto a digit and Berstein method fails to capture accuracy upto 3 digits even for N = 6. Thus, the fruitfulness of the present method is enhanced in this table.

Example 4.8. *Let us consider* (2.1) *with* $m = 3$, $m_1 = 0$, $m_2 = 0$, $p_{02} = 1$, $\alpha_{02} = 1$, $h_{012} = 1$, $h_{013} = -1$,$\alpha_{012} = 0.1$, $\alpha_{013} = \frac{1}{8}$, $\gamma_{012} = 0.1$, $\gamma_{013} = 0.1$ *which is a third order nonlinear pantograph equation with constant coefficient given by:*

$$w^{(3)}(t) = w^{(2)}(t) + w^{(2)}(0.1t)w^{(1)}(0.1t) - w^{(3)}(0.1t)w^{(1)}(\frac{t}{8}) - f(t), \tag{4.11}$$

with initial conditions $w(0) = 1$, $w^{(1)}(0) = 0$, $w^{(2)}(0) = -1$. *The exact solution is given by* $w(t) = cos(t)$. *The* $f(t)$ *can be calculated on substituting the exact solution into the equation.*

Table 10 tabulates the absolute error obtained using present method for $N = 8$. The table shows that absolute error obtained by present method shows accuracy of order $O(10^{-9})$. This equation is not attempted in the literature yet.

Table 11. Comparison of absolute error for Example 9 with SOB-Operational [13], FHBF [4] and BW [12].

t	SOB- [13]	FHBF [4]	Present Method at $N = 8$			BW [12]	Present method
	$N = 8$	$N = 8$	Chebyshev	Legendre	Jacobi	$N = 14$	$N = 14$
0.2	$6.78 * 10^{-11}$	$3.05 * 10^{-7}$	$6.89261 * 10^{-12}$	$4.248984 * 10^{-12}$	$5.052381 * 10^{-12}$	$2.332 * 10^{-15}$	$1.05163 * 10^{-19}$
0.4	$3.31 * 10^{-10}$	$9.81 * 10^{-7}$	$1.74226 * 10^{-13}$	$5.298887 * 10^{-12}$	$1.748448 * 10^{-11}$	$1.055 * 10^{-14}$	$4.8761 * 10^{-19}$
0.6	$8.43 * 10^{-10}$	$1.85 * 10^{-6}$	$1.00156 * 10^{-10}$	$7.78217 * 10^{-11}$	$7.1783491 * 10^{-12}$	$2.465 * 10^{-14}$	$1.1491 * 10^{-18}$
0.8	$1.54 * 10^{-9}$	$2.64 * 10^{-6}$	$1.75342 * 10^{-10}$	$7.300290 * 10^{-11}$	$1.589979 * 10^{-11}$	$4.452 * 10^{-14}$	$2.0784 * 10^{-18}$
1	$2.49 * 10^{-9}$	$3.03 * 10^{-6}$	$2.54386 * 10^{-8}$	$2.66333 * 10^{-8}$	$2.771830 * 10^{-8}$	$6.950 * 10^{-14}$	$8.5416 * 10^{-18}$
CPU Time	1.1388		0.016				

Table 12. Comparison of absolute error for Example 4.10 with JRGCM [5].

t	JRGCM [5] at $N = 12$		BW [12]	Present Method at $N = 12$		
	$\alpha = \beta = \frac{3}{2}$	$\alpha = \beta = \frac{1}{2}$	$N = 12$	Chebyshev	Legendre	Jacobi
0.1	$1.259 * 10^{-6}$	$1.389 * 10^{-6}$	$3.389 * 10^{-10}$	$8.44875 * 10^{-13}$	$2.20335 * 10^{-12}$	$6.35201 * 10^{-12}$
0.3	$3.703 * 10^{-5}$	$3.999 * 10^{-5}$	$7.094 * 10^{-9}$	$2.18008 * 10^{-11}$	$6.15971 * 10^{-11}$	$2.01928 * 10^{-10}$
0.5	$1.760 * 10^{-4}$	$1.811 * 10^{-4}$	$3.468 * 10^{-8}$	$2.70294 * 10^{-10}$	$4.49383 * 10^{-10}$	$1.12922 * 10^{-9}$
0.7	$5.247 * 10^{-4}$	$5.510 * 10^{-4}$	$8.196 * 10^{-8}$	$1.77315 * 10^{-8}$	$1.64922 * 10^{-8}$	$1.58143 * 10^{-8}$
0.9	$1.693 * 10^{-3}$	$1.718 * 10^{-3}$	$3.794 * 10^{-6}$	$3.80630 * 10^{-7}$	$3.42334 * 10^{-7}$	$2.86995 * 10^{-7}$

Example 4.9. *Let us consider* (2.1) *with* $m = 3$, $m_1 = 1$, $p_{00} = -1$, $\alpha_{00} = 1$, $p_{10} = -1$, $\alpha_{10} = 1$, $\beta_{10} = -0.3$, $f(t) = e^{-t+0.3}$ *which is a third order pantograph equation with time varying delay equation given by [13]:*

$$w^{(3)}(t) = -w(t) - w(t - 0.3) + e^{-t+0.3}, \tag{4.12}$$

with initial conditions $w(0) = 1$, $w^{(1)}(0) = -1$, $w^{(2)}(0) = 1$. *The exact solution is given by* $w(t) = e^{-t}$.

Table 11 compares the absolute error with Shifted Orthogonal Berstein Operational method (SOB-Operational)[13], FHBFs [4] and Bernoulli Wavelets (BW) [12] for different values of N. The table shows that the present method shows better accuracy compared to other methods with less computational time.

Example 4.10. *Let us consider* (2.1) *with* $m = 4$, $m_1 = 0$, $p_{00} = -1$, $\alpha_{00} = 0.5$, $p_{01} = -1$, $\alpha_{01} = 1$, $p_{02} = -t$, $\alpha_{02} = 2$, $p_{03} = 1$, $\alpha_{03} = 0.25$ *which is a fourth order pantograph differential equation with variable coefficient [5]:*

$$w^{(4)}(t) = w^{(3)}(0.25t) - tw^{(2)}(2t) - w^{(1)}(t) - w(0.5t) - f(t), \tag{4.13}$$

with the initial conditions $w(0) = 0$, $w^{(1)}(0) = 2$, $w^{(2)}(0) = -4$, $w^{(3)}(0) = -2$. *The exact solution is given by* $w(t) = e^{-t}sin(2t)$. *The* $f(t)$*is given by substituting the exact solution in* (4.13).

Table 12 compares the absolute error obtained using present method with those of Jacobi Rational- Gauss Collocation Method (JRGCM) [5] and Bernoulli Wavelets (BW) [12] for $N = 12$. This table reveals that MAE obtained by present method shows two times better accuracy for same number of collocation points compared to JRGCM. (i.e.) For $N = 12$, present method shows MAE of $O(10^{-7})$, whereas JRGCM shows MAE of $O(10^{-3})$ only. The table also shows that the considered special functions works well compared to Bernoulli functions.

Example 4.11. *Let us consider* (2.1) *with* $m = 5$, $m_1 = 0$, $p_{00} = 1$, $\alpha_{00} = 0.5$, $p_{01} = 1$, $\alpha_{01} = 2$,$p_{02} = -3$, $\alpha_{02} = 0.25$, $p_{03} = 2t$, $\alpha_{03} = \frac{1}{3}$, $p_{04} = -1$, $\alpha_{04} = 1$ *which is a fifth order pantograph differential equations with variable coefficients given by [5]:*

$$w^{(5)}(t) + w^{(4)}(t) = 2tw^{(3)}(\frac{t}{3}) - 3w^{(2)}(0.25t) + w^{(1)}(2t) + w(0.5t) - f(t), \tag{4.14}$$

TABLE 13. Comparison of absolute error for Example 4.11 with JRGCM [5].

t	JRGCM [5] at $N = 18$			Present Method at $N = 10$		
	$\alpha = \beta = 1$	$\alpha = \beta = 0.5$	$\alpha = \beta = 0.5$	Chebyshev	Legendre	Jacobi
0.1	$4.149 * 10^{-6}$	$3.676 * 10^{-6}$	$1.221 * 10^{-5}$	$6.71228 * 10^{-8}$	$1.06874 * 10^{-8}$	$2.51323 * 10^{-8}$
0.3	$5.872 * 10^{-4}$	$1.186 * 10^{-4}$	$9.244 * 10^{-4}$	$9.70219 * 10^{-7}$	$8.51033 * 10^{-7}$	$2.54472 * 10^{-6}$
0.5	$4.465 * 10^{-3}$	$1.021 * 10^{-3}$	$7.084 * 10^{-3}$	$4.544170 * 10^{-5}$	$6.37292 * 10^{-5}$	$1.97627 * 10^{-5}$
0.7	$1.606 * 10^{-2}$	$3.257 * 10^{-3}$	$2.515 * 10^{-2}$	$1.422850 * 10^{-5}$	$2.37034 * 10^{-5}$	$7.46599 * 10^{-5}$
0.9	$4.397 * 10^{-2}$	$1.023 * 10^{-2}$	$6.712 * 10^{-2}$	$1.45385 * 10^{-4}$	$7.23294 * 10^{-5}$	$2.09096 * 10^{-4}$

TABLE 14. Comparison of absolute error for Example 4.12 with BCM [11].

t	Berstein Series [11]	Present Method at $N = 8$		
	N = 8	Chebyshev	Legendre	Jacobi
0.2	$4.549 * 10^{-6}$	$2.03446 * 10^{-11}$	$4.19300 * 10^{-11}$	$9.43293 * 10^{-11}$
0.4	$1.352 * 10^{-5}$	$2.51095 * 10^{-10}$	$6.50428 * 10^{-10}$	$1.72196 * 10^{-9}$
0.6	$6.844 * 10^{-4}$	$1.33049 * 10^{-9}$	$3.36646 * 10^{-9}$	$9.14699 * 10^{-9}$
0.8	$3.517 * 10^{-3}$	$4.94116 * 10^{-9}$	$1.16149 * 10^{-8}$	$3.07055 * 10^{-8}$
1	$1.022 * 10^{-2}$	$7.10138 * 10^{-8}$	$8.39185 * 10^{-8}$	$1.25749 * 10^{-7}$

with the initial conditions $w(0) = 1$, $w^{(1)}(0) = \frac{-1}{2}$, $w^{(2)}(0) = \frac{-63}{4}$, $w^{(3)}(0) = \frac{191}{8}$, $w^{(4)}(0) = \frac{3713}{16}$. *The exact solution is given by* $w(t) = e^{-2t}cos(4t)$. *The solution* $f(t)$*is obtained by substituting exact solution in* (4.14).

Table 13 compares the absolute error obtained using present method for $N = 10$ with those of JRGCM[5] for $N = 18$ (for different values of α and β). This table shows that MAE for present method is of $O(10^{-4})$, whereas for JRGCM, MAE goes upto $O(10^{-2})$. Though [5] is studied using Jacobi polynomials, due to the efficiency of present method, a high accuracy of results are obtained compared to [5].

Example 4.12. *Let us consider* (2.1) *with* $m = 5$, $m_1 = 0$, $p_{00} = -1$, $\alpha_{00} = 1$, $\beta_{00} = -0.3$, $p_{01} = 1$, $\alpha_{01} = 1$, $p_{02} = -t$, $\alpha_{02} = 1$, $p_{04} = t$, $\alpha_{04} = 1$, $f(t) = e^{(-t+0.3)}$ *which is a variable fifth order time varying delay pantograph differential equation given by [11]:*

$$w^{(5)}(t) = tw^{(4)}(t) - tw^{(2)}(t) + w^{(1)}(t) - w(t - 0.3) + f(t), \tag{4.15}$$

with initial conditions $w(0) = 1$, $w^{(1)}(0) = -1$, $w^{(2)}(0) = 1$, $w^{(3)}(0) = -1$, $w^{(4)}(0) = 1$. *The exact solution is given by* $w(t) = e^{-t}$.

Table 14 compares the absolute error obtained using present method with Berstein Collocation Method (BCM) [11]for $N = 8$. It can be heeded from the table, the present method obtain MAE of $O(10^{-7})$, whereas BCM shows MAE of $O(10^{-2})$ for same number of collocation points. The significance of the present method of acquiring better accuracy compared to the other method is captured in this table.

5. CONCLUSION

This paper proposes the spectral solution for the framed generalized form of higher order linear/nonlinear delay differential equations. Spectral solutions are presented for twelve different existing examples of higher order linear/nonlinear delay differential equations. The absolute and residual errors obtained for these examples using the present method are compared with the available literature works. The following conclusions are drawn from the present analysis.

(i) It may be noted from the error tables and figures that the present method prevails over many other studies in the literature in yielding high accuracy at the same number of collocation points. Hence this study gains more importance.

(ii) The significance of this method lies in showing the robustness and computational effectiveness of the present method. (i.e) the CPU time used by the present method is less compared to other methods which are shown in tables.

(iii) In all the examples, there occurs a decline in the absolute and residual error with raise in N. (i.e) The error converges.

(iv) Though, there are wide collections of collocation methods available in the literature, the comparative tables and figures picturise that the present method shows much better results than the previously available collocation methods.

(v) All these tables and figures evidently reveal that the present method is more suitable to solve higher order linear/ nonlinear delay differential equations and yields high accuracy at the low number of collocation points and computational time.

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