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Solving The Stefan Problem with Kinetics

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Abstract We introduce and discuss the Homotopy perturbation method, the Adomian decomposition method and the variational iteration method for solving the stefan problem with kinetics. Then, we give an example of the stefan problem with kinetics and solve it by these methods.

Keywords. Stefan problem, Kinetics, Homotopy perturbation, Adomian decomposition, Variational iteration.

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1. Introduction

Phase-changes, or the Stefan problems in which material melts or solidifies occur in a wide variety of natural and industrial processes. Mathematically, these are special cases of moving-boundary problems, in which the location of the front between the solid and liquid is not known beforehand, but must be determined as a part of the solution [8]. The basic partial differential equation is heat transfer equation, nevertheless, solving the problem is not straightforward due to the moving boundary. In general, when solving the problem, the technique should be able to track the moving boundary. Stefan problems model, many real world and engineering situations [16, 46]. Examples include solidification of metals, freezing of water and food, crystal growth, casting, welding, melting, ablation, etc. Many numerical methods have been used for solving the Stefan problems. Crank [8] as well as Lynch and ONeill [38] provide a comprehensive summary of the numerical methods used for this type of problems. Phase-change problems have always remained an active area of research. Analytical progress in the solution of Stefan problems has remained very limited and usually unavailable. In one-dimensional Stefan problem we wish to determine the free boundary (sufficiently smooth) which is given by $x = s(t)$ and the temperature solution $u(x,t)$. In this paper, we consider the modified one-phase Stefan problem and seek a solution $(u(x,t), s(t))$, which satisfies the one-dimensional heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - \gamma u(x,t), \quad -\infty < x < s(t)\ , t > 0, \tag{1.1}$$

subject to the initial condition

$$u(x,0) = \varphi(x), \tag{1.2}$$

and other conditions on the moving boundary $x = s(t)$ as

$$\frac{\partial u}{\partial x}(s(t),t) = -V(t), \quad t \in (0,T], \tag{1.3}$$

$$g(u(s(t),t)) = V(t), \quad t \in (0,T], \tag{1.4}$$

in which the damping term $\gamma \geq 0$ is subject to volumetric heat losses, $V(t) = s'(t)$ is the velocity, $s(0) = 0$. Further assume that $g(t)$ is monotonically decreasing differentiable function on $[0,\infty)$ with $|g'(t)| \leq C$ which satisfies

$$-V_0 \leq g(t) \leq -v_0, \quad for\ some\ V_0, v_0 > 0. \tag{1.5}$$

The above problem arises naturally as a mathematical model of a variety of exothermic phase transition type processes, such as condensed phase combustion [39] also known as self-sustained high temperature synthesis or SHS [41], solidification with undercooling [35], laser induced evaporation [15], rapid crystallization in thin films [50] etc. Existence and uniqueness of bounded classical solutions for the problem (1.2) − (1.4) was established in [12].

The paper is organized as follows: Section 2 introduces the homotopy perturbation method, the Adomian decomposition method and the variational iteration method. Section 3 is devoted to present the application of these methods to the Stefan problem. Finally in section 4, two numerical examples are given to demonstrate the accuracy of the methods.

2. Description of Methods

In this section we will briefly discuss the homotopy perturbation, the Adomian decomposition and the variational iteration methods.

2.1. **Homotopy Perturbation Method (HPM).**

Homotopy perturbation method was first proposed by He [18]. The method is a powerful and efficient tool for finding solutions of linear and non-linear equations. It has been used to obtain the solutions of a large class of linear and non-linear equations [19, 20, 21, 23, 24, 25]. To illustrate the basic ideas of this method, we consider the following non-linear functional equation:

$$A(u) - f(r) = 0, \ r \in \Omega, \tag{2.1}$$

with the following boundary condition:

$$B(u, \frac{\partial u}{\partial n}) = 0, \ r \in \Gamma, \tag{2.2}$$

where A is a general functional operator, B is a boundary operator, $f(r)$ is a known analytical function and Γ is the boundary of the domain Ω. The operator A can be decomposed into two operators L and N, where L is linear, and N is nonlinear. Equation (2.1) can be, therefore, written as follows:

$$L(u) + N(u) - f(r) = 0. \tag{2.3}$$

Using the homotopy technique, we construct a homotopy $v(r,p) : \Omega \times [0,1] \to \mathbf{R}$ which satisfies:

$$H(v,p) = (1-p)[L(v) - L(u_0)] + p[A(v) - f(r)] = 0, \ p \in [0,1], \ r \in \Omega, \quad (2.4)$$

or equivalently,

$$H(v,p) = L(v) - L(u_0) + pL(u_0) + p[N(v) - f(r)] = 0. \quad (2.5)$$

where $p \in [0,1]$ is an embedding parameter and u_0 is an initial approximation for the solution of equation (2.1) which satisfies the boundary conditions. Obviously, from Equations (2.3) and (2.4) we will have:

$$H(v,0) = L(v) - L(u_0) = 0, \quad (2.6)$$

$$H(v,1) = A(u) - f(r) = 0. \quad (2.7)$$

The changing values of p from zero to unity are just that of $u(r,p)$ from $u_0(r)$ to $u(r)$. In topology, this is called homotopy. According to HPM, we can first use the embedding parameter p as a small parameter,and assume that the solution of Equations (2.3) and (2.4) are power series in p:

$$v = v_0 + pv_1 + p^2 v_2 + \cdots, \quad (2.8)$$

Setting $p = 1$ results in the approximation to the solution of Equation (2.1) as

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + \cdots. \quad (2.9)$$

The combination of the perturbation method and the homotopy method is called the homotopy perturbation method, which has eliminated limitations of the traditional perturbation techniques. The series (2.9) is convergent for many cases. Some criteria are suggested for convergence of the series (2.9) in [18].

2.2. **Analysis of the Adomian Decomposition Method (ADM).** In the early 1980s, a new numerical method was developed by George Adomian [2] in order to solve non-linear functional equations of the form

$$Lu + Ru + Nu = g, \quad (2.10)$$

using an iterative decomposition scheme that led to elegant computation of closed-form analytical solutions or analytical approximations to solutions. ADM can excellently treat a wide variety of functional equations [3, 4, 9, 10, 11, 14, 17, 34, 36, 47]. In (2.10), operator L represents the linear part, operator R represents the remainder or lower order terms, operator N represents the non-linear part and g is the non-homogeneous right hand side. The solution u and the non-linearity N are assumed to have respectively, the following analytic expansions,

$$u = \sum_{n=0}^{\infty} u_n, \quad Nu = \sum_{n=0}^{\infty} A_n, \quad (2.11)$$

where the A_n' s are the Adomian polynomials that depend only on u_0, u_1, $\cdots$, u_n given by the following formula:

$$A_n(u_0, u_1, \cdots, u_n) = \frac{1}{n!}\frac{d^n}{d\lambda^n}(N\sum_{n=0}^{\infty} u_k\lambda^k)|_{\lambda=0},\ n = 0, 1, 2, \cdots . \tag{2.12}$$

In order to better explain the method, we will first assume the convergence of the series in (2.11) and deal with the rigorous convergence issues later. The parameter k is a dummy variable introduced for ease of computation. There are several different versions of (2.12) that can be found in the literature that leads to easier computation of the A_n's. It should be noted that the A_n's are the terms of analytic expansion of $N\hat{u}$, where $\hat{u} = \sum_{n=0}^{\infty} u_n\lambda^n$ about $\lambda = 0$ [13]. In [2] Adomian has shown that the expansion for Nu in (3.2) is a rearrangement of the Taylor series expansion of Nu about the initial function u_0 in a suitable Hilbert or Banach space. Substitution of (2.11) in (2.10) results in the following:

$$L(\sum_{n=0}^{\infty} u_n) = -R(\sum_{n=0}^{\infty} u_n) - \sum_{n=0}^{\infty} A_n + g. \tag{2.13}$$

The above equation can be rewritten in a recursive fashion, yielding iterates of u_n, the sum of which converges to the solution u satisfying (2.13) if it exists:

$$\begin{aligned} \sum_{n=0}^{\infty} u_n &= \phi - L^{-1}R(\sum_{n=0}^{\infty} u_n) - \sum_{n=0}^{\infty} L^{-1}A_n + L^{-1}g, \\ u_0 &= \phi + L^{-1}g, \\ u_{n+1} &= -L^{-1}R(u_n) - L^{-1}A_n. \end{aligned} \tag{2.14}$$

Typically, the symbol L^{-1} represents a formal inverse of the linear operator L. In the case of partial differential equations, L is the highest order partial derivative operator for which the formal inverse can be computed using integrations. A general theory of decomposition schemes for non-linear functional equations was developed by Gabet [13]. Convergence results as applied to ordinary differential equations and non-linear functional equations can be found in [1, 6, 7]. Mavoungou [40] has proved a convergence result for the Adomian scheme as applied to partial differential equations. A compendium of interesting examples of partial differential equations for which the ADM was utilized can be found in [49]. In general, the iterates in the Adomian decomposition scheme (2.14) converge very rapidly to the unique solution of the functional equation (2.10) provided that the scheme satisfies the property of strong convergence as discussed in [13].

2.3. **Variational Iteration Method (VIM).** To illustrate the basic concept of He's VIM, we consider the following general differential equation:

$$Lu + Nu = g(x), \tag{2.15}$$

where L is a linear operator, N is a nonlinear operator, and $g(x)$ is the inhomogeneous term. According to variational iteration method [26, 27, 28, 29, 30, 31, 32, 33, 42, 43, 44, 45], we can construct a correction functional as follows:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(Lu_n(s) + N\widetilde{u}_n(s) - g(s))ds, \tag{2.16}$$

where λ is a Lagrange multiplier [26, 27, 28, 29], which can be identified optimally via VIM. The subscripts n denote the nth approximation, and $\widetilde{u}_n$ is considered to be a restricted variation. That is $\delta\widetilde{u}_n = 0$. The solution of the linear problems can be solved in a single iteration step due to the exact identification of the Lagrange multiplier. The principles of VIM and its applicability for various kinds of differential equations are given in [26, 27, 28, 29]. In this method, it is required first to determine the Lagrange multiplier λ optimally. The successive approximation u_{n+1}, $n \geq 0$, of the solution u will be readily obtained upon using the determined Lagrange multiplier and any selective function u_0; consequently, the solution is given by $u = \lim_{n\to\infty} u_n$. The convergence of variational iteration method has been discussed in [48].

3. Applications

In this subsection, application of all these methods to the Stefan problem with kinetics is briefly described. Substituting $v = \exp(\gamma t)u$, in $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - \gamma u(x,t)$ yields $v_t = v_{xx}$ since

$$v_t = \gamma \exp(\gamma t)u + \exp(\gamma t)u_t = \exp(\gamma t)(\gamma u + u_t) = v_{xx}, \tag{3.1}$$

hence we can put $\gamma = 0$.

3.1. **Application of HPM.** In this section, we will apply the homotopy perturbation method to the free boundary problem (1.1)-(1.4). According to the HPM, we construct the following simple homotopy for the equation (1.1)

$$\begin{aligned} H(v,p) &= (1-p)v_t + p(v_t + v_{xx}) \\ &= v_t + pv_{xx} = 0, \end{aligned} \tag{3.2}$$

where $p \in [0,1]$ is an embedding parameter. When $p = 0$, (3.2) is an ordinary differential equation with $v_t = 0$ which is easy to solve; and if, $p = 1$ it turns out to be the equation (1.1). The basic assumption is that the solution can be written as a power series in p as

$$v = v_0 + pv_1 + p^2 v_2 + p^3 v_3 + \ldots = \sum_{i=0}^{\infty} p^i v_i. \tag{3.3}$$

Now, by substituting (3.3) in (3.2), we get,

$$(v_0)_t + p(v_1)_t + p^2(v_2)_t + \ldots - p((v_0)_{xx} + p(v_1)_{xx} + p^2(v_2)_{xx} + \ldots) = 0. \tag{3.4}$$

In order to obtain the coefficients of various powers of p , we need to compare the different powers of p. Since resulting equations are ordinary differential equations of first order, we need an initial condition for any such equation. Comparing the terms with identical powers of p in (3.4) together with the initial condition (1.2), the following series of equations can be obtained

$$\begin{aligned} p^0: \quad & (v_0)_t = 0 \\ & v_0(x,0) = \varphi(x), \end{aligned} \tag{3.5}$$

$$p^i: \quad (v_i)_t = (v_{i-1})_{xx}, \; for \; i = 1, 2, ..., \qquad v_i(x, 0) = 0. \tag{3.6}$$

By taking the limit of $v(x, t)$, we can obtain the solution of the free boundary problem as

$$u(x, t) = \lim_{p \to 1}(v_0 + pv_1 + p^2 v_2 + ...). \tag{3.7}$$

Now, using the relation (1.3), one can simply compute the free boundary $s(t)$. Using the initial condition $s(0) = 0$ and the condition (1.3), we can construct the following ordinary differential equation

$$s'(t) = F(t, s(t)), \qquad s(0) = 0, \tag{3.8}$$

where $F(t, s(t)) = -\frac{\partial u}{\partial x}(s(t), t)$. Solving this ODE, one can get the free boundary function $s(t)$. Finally, we need to show that the resulting solution $(u(x, t), s(t))$ satisfies the condition (1.4).

3.2. **Application of ADM.** To convey the process of solving the Stefan problem with kinetics $(1.1) - (1.4)$ by Adomian decomposition method, we apply the inverse operator, $L^{-1} = \int_0^t (.) d\tau$, to the heat equation (1.1),

$$\begin{aligned} u(x, t) &= u(x, 0) + \int_0^t u_{xx}(x, \tau) d\tau, \\ &= \varphi(x) + \int_0^t u_{xx}(x, \tau) d\tau. \end{aligned} \tag{3.9}$$

By using (2.14), we can write,

$$\begin{aligned} &u_0 = \varphi(x), \\ &u_{n+1}(x, t) = \int_0^t (u_n)_{xx}(x, \tau) d\tau. \end{aligned} \tag{3.10}$$

We can obtain the solution of the free boundary problem as

$$u(x, t) = \sum_{n=0}^{\infty} u_n. \tag{3.11}$$

To compute the free boundary $s(t)$, we apply the same approach used in HPM above. Finally, we need to show that the above solution (u, s) satisfies the condition (1.4).

3.3. **Application of VIM.** In this subsection, the variational iteration method is applied for solving problems $(1.1) - (1.4)$. According to the variational iteration method, we consider the correction functional for (3.1) in the form:

$$u_{n+1}(x, t) = u_n(x, t) + \int_0^t \lambda(s)(\frac{\partial u_n}{\partial s} - \frac{\partial^2 \widetilde{u}_n}{\partial x^2}) ds. \tag{3.12}$$

where λ is the general Lagrange multiplier, u_0 is an initial approximation which should be chosen suitably and $\widetilde{u}_n$ is the restricted variation, i.e. $\delta \widetilde{u}_n = 0$. To find the optimal value of λ we have

$$\delta u_{n+1}(x, t) = \delta u_n(x, t) + \delta \int_0^t \lambda(s)(\frac{\partial u_n}{\partial s} - \frac{\partial^2 \widetilde{u}_n}{\partial x^2}) ds, \tag{3.13}$$

or

$$\delta u_{n+1}(x,t) = \delta u_n(x,t)(1+\lambda) - \delta \int_0^t u_n \frac{\partial \lambda(s)}{\partial s} ds = 0, \tag{3.14}$$

which yields

$$\frac{\partial \lambda(s)}{\partial s} = 0, \quad 1+\lambda = 0. \tag{3.15}$$

Thus we have $\lambda = -1$, and we obtain the following iteration formula:

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t (\frac{\partial u_n}{\partial s} - \frac{\partial^2 u_n}{\partial x^2}) ds, \tag{3.16}$$

and for sufficiently large values of n we can consider u_n as an approximation of the exact solution. Using the same approach as HPM, we can compute the free boundary function $s(t)$.

4. Numerical Results

Now, we are ready to apply the HPM and ADM to the calculation of the Stefan problem through the test by numerical examples.

Example 1. Let $\alpha \in \mathbb{R}$ be positive. Consider the problem $(1.1) - (1.4)$ with the following data:

$$\begin{aligned} \varphi(x) &= \exp(\alpha x), \\ g(t) &= \exp(-t) - \alpha - \tfrac{1}{e}, \end{aligned} \tag{4.1}$$

which correspond to the exact solution

$$\begin{aligned} u(x,t) &= \exp(\alpha^2 t + \alpha x), \\ s(t) &= -\alpha t. \end{aligned} \tag{4.2}$$

Precisely, $g(t)$ is monotonically decreasing differentiable function on $[0,\infty)$ with $|g'(t)| = e^{-t} \leq 1$ and g satisfies the assumption (1.5).

Case a: HPM

As mentioned in the previous section, from relations (3.5) and (4.1) we get

$$\begin{aligned} p^0: \quad & (v_0)_t = 0 \\ & v_0(x,0) = \exp(\alpha x). \end{aligned} \tag{4.3}$$

Solving this boundary value problem yields,

$$v_0 = \exp(\alpha x). \tag{4.4}$$

For the first exponent of p by (3.5) we can write

$$\begin{aligned} p^1: \quad & (v_1)_t = (v_0)_{xx} = \alpha^2 \exp(\alpha x), \\ & v_1(x,0) = 0. \end{aligned} \tag{4.5}$$

The solution of this problem is

$$v_1 = \alpha^2 \exp(\alpha x) t. \tag{4.6}$$

By a similar argument we obtain,

$$\begin{aligned} v_2 &= \alpha^4 \exp(\alpha x)\tfrac{t^2}{2}, & v_3 &= \alpha^6 \exp(\alpha x)\tfrac{t^3}{3!}, \\ v_4 &= \alpha^8 \exp(\alpha x)\tfrac{t^4}{4!}, & v_5 &= \alpha^{10} \exp(\alpha x)\tfrac{t^5}{5!}. \end{aligned} \tag{4.7}$$

Now, we can calculate the solution of the above Stefan problem as

$$\begin{aligned} u(x,t) &= \lim_{p\to 1}(v_0 + pv_1 + p^2 v_2 + \ldots) \\ &= \lim_{p\to 1}\sum_{k=0}^{\infty} \exp(\alpha x) p^k \frac{(\alpha^2 t)^k}{k!} \\ &= \sum_{k=0}^{\infty} \exp(\alpha x) \frac{(\alpha^2 t)^k}{k!} \\ &= \exp(\alpha x)\exp(\alpha^2 t) = \exp(\alpha x + \alpha^2 t). \end{aligned} \tag{4.8}$$

By using (1.3) and the above relation we have

$$s'(t) = -V(t) = -u_x(s(t),t) = -\alpha \exp(\alpha s(t) + \alpha^2 t). \tag{4.9}$$

Then

$$\int_0^t -s'(\tau)\exp(-\alpha s(\tau))d\tau = \int_0^t \alpha \exp(\alpha^2 \tau)d\tau. \tag{4.10}$$

Note that $s(0) = 0$. Using this fact, the solution of the above integral is

$$s(t) = -\alpha t. \tag{4.11}$$

This (u, s) is the exact solution of the Stefan problem corresponding to the data (4.1). Now, we can simply show that this solution also satisfies the conditions (1.4), as follows

$$\begin{aligned} u(s(t),t) &= \exp(-\alpha^2 t + \alpha^2 t) = 1, \\ g(u(s(t),t)) &= g(1) = \exp(-1) - \alpha - \tfrac{1}{e} = -\alpha = V(t). \end{aligned} \tag{4.12}$$

Case b: ADM

By relations (3.9) we have

$$\begin{aligned} &u_0 = \exp(\alpha x), \\ &u_1(x,t) = \textstyle\int_0^t (u_0)_{xx} d\tau = \exp(\alpha x)\frac{\alpha^2 t}{1!}, \\ &u_2(x,t) = \textstyle\int_0^t (u_1)_{xx} d\tau = \exp(\alpha x)\frac{(\alpha^2 t)^2}{2!}, \\ &\vdots \\ &u_n(x,t) = \textstyle\int_0^t (u_{n-1})_{xx} d\tau = \exp(\alpha x)\frac{(\alpha^2 t)^n}{n!}, \\ &\vdots \end{aligned} \tag{4.13}$$

By computing $u(x,y)$ from (3.10) we obtain

$$\begin{aligned} u(x,t) &= \lim_{n\to\infty}\sum_{k=0}^{n} u_k = \exp(\alpha x)\sum_{n=0}^{\infty}\frac{(\alpha^2 t)^n}{n!} \\ &= \exp(\alpha x+\alpha^2 t). \end{aligned} \tag{4.14}$$

By a similar argument to the case of HPM, one can conclude that:

$$\begin{aligned} &s(t) = -\alpha t, \\ &g(u(s(t),t)) = g(1) = \exp(-1) - \alpha - \tfrac{1}{e} = -\alpha = V(t). \end{aligned} \tag{4.15}$$

This solution is the exact solution of the inhomogeneous heat equation and by (4.12) satisfies the conditions (1.3) – (1.4).

Case c: VIM

Using the variational iteration method for solving the Stefan problem, we can select $u_0(x,t) = \exp(\alpha x)$ by using the given initial value. Accordingly, by the iteration formula (3.15), we obtain the following successive approximations:

$$\begin{aligned} &u_1(x,t) = \exp(\alpha x)[1+(\alpha^2 t)], \\ &u_2(x,t) = \exp(\alpha x)[1+(\alpha^2 t)+\tfrac{(\alpha^2 t)^2}{2!}], \\ &\vdots \\ &u_n(x,t) = \exp(\alpha x)[1+(\alpha^2 t)+\tfrac{(\alpha^2 t)^2}{2!}+\cdots+\tfrac{(\alpha^2 t)^n}{n!}] = \exp(\alpha x)\sum_{k=0}^{n}\frac{(\alpha^2 t)^n}{n!}, \\ &\vdots \end{aligned} \tag{4.16}$$

Recall that

$$u(x,t) = \lim_{n\to\infty} u_n(x,t). \tag{4.17}$$

Consequently, the exact solution is the form:

$$u(x,t) = \exp(\alpha x+\alpha^2 t). \tag{4.18}$$

From Equations (4.8) and (4.14) we see that the approximate solution of the problem (1.1) – (1.4) obtained by using the variational iteration method is the same as the one obtained by the homotopy perturbation method and by the Adomian decomposition method, it seems that the approximate solution remains closer to exact solution. By a similar argument to the that cases of HPM and ADM, one can conclude that

$$\begin{aligned} &s(t) = -\alpha t, \\ &g(u(s(t),t)) = g(1) = \exp(-1) - \alpha - \tfrac{1}{e} = -\alpha = V(t). \end{aligned} \tag{4.19}$$

Example 2. In this example we choose the data as $\varphi(x) = x^2+x$, $g(t) = \frac{1}{10^{10}}(\frac{1}{t+1} - 2)$. Clearly the function g satisfy the assumptions stated in the introduction and (1.5). Moreover this problem has no exact solution.

Case 1: HPM. Using the relations (3.5) and (3.6) one can get

$$\begin{aligned} v_0 &= x^2 + x, \\ v_1 &= 2t, \\ v_n &= 0, \; n = 2, 3, \cdots . \end{aligned} \tag{4.20}$$

Using (3.7), we can compute the solution $u(x,t)$

$$u(x,t) = \lim_{p\to 1}(v_0 + pv_1 + p^2 v_2 + \cdots) = \lim_{p\to 1}(v_0 + pv_1). \tag{4.21}$$

Then we get

$$u(x,t) = x^2 + x + 2t. \tag{4.22}$$

To compute the free boundary, we apply the relation (3.8)

$$\begin{aligned} s'(t) &= -(2s(t) + 1), \\ s(0) &= 0. \end{aligned} \tag{4.23}$$

Solving this ODE yields $2s(t) + 1 = e^{-2t}$. For the condition (1.4), we have

$$\begin{aligned} |s'(t) - g(u(s(t),t))| &= | - e^{-2t} \\ &\quad - \frac{1}{10^{10}}(\frac{1}{\frac{1}{4}(exp(-2t) - 1)^2 + \frac{1}{2}(exp(-2t) - 1) + 2t + 1} - 2)| \\ &\leq 2 \times 10^{-10}, \; for\ sufficiently\ large\ t. \end{aligned} \tag{4.24}$$

Case 2: ADM. To obtain $u(x,t)$ in this case using (3.10) one can get

$$\begin{aligned} u_0(x,t) &= x^2 + x, \\ u_1(x,t) &= 2t, \\ u_n(x,t) &= 0, \; n = 2, 3, \cdots . \end{aligned} \tag{4.25}$$

Thus the computation of the solution of the free boundary problem using (3.11) yields

$$u(x,t) = u_0(x,t) + u_1(x,t) = x^2 + x + 2t. \tag{4.26}$$

Applying the same approach used above in the case of HPM, we can get the free boundary function

$$2s(t) + 1 = e^{-2t}. \tag{4.27}$$

At last in this case the same relation as (4.24) holds for the condition (1.4).

Case 3: VIM. In this case we select $u_0 = x^2 + x$ and to compute u_n for $n = 1, 2, \cdots$ using the recursive relation (3.16), we can obtain

$$u_n(x,t) = x^2 + x + 2t, \; n = 1, 2, \cdots . \tag{4.28}$$

Taking the limit in the above nth relation and remembering $u = \lim_{n\to\infty} u_n$, one can get the solution

$$u(x,t) = x^2 + x + 2t. \tag{4.29}$$

Applying the same approach as in the case of HPM, we can get the free boundary function

$$2s(t) + 1 = e^{-2t}. \tag{4.30}$$

For the condition (1.4) in this case the relation (4.24) also holds.

5. Conclusion

In this paper, Homotopy perturbation, Adomian decomposition and variational iteration methods are employed successfully to study the Stefan problem with kinetics. As it is seen, all these methods obtain the exact solution or get the good approximate solution of the Stefan problem with kinetics. Moreover, they are straightforward and avoid the hectic work of calculations. In conclusion, these three methods provide highly accurate numerical solutions for free boundary problems. As it is mentioned, these methods avoid physically unrealistic assumptions. These method are applied in a direct way without using linearization, transformation, discretization or restrictive assumptions. Variational iteration method gives several successive approximations through the iteration of the correction functional. Finally, comparison with exact solutions reveals that HPM, ADM and VIM are remarkably effective for solving these types of problems. Authors believe that these methods provide efficient techniques for solving various scientific and engineering problems.

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