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A Bernoulli Tau method for numerical solution of feedback Nash differential games with an error estimation

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Abstract

In the present study, an efficient combination of the Tau method with the Bernoulli polynomials is proposed for computing the Feedback Nash equilibrium in differential games over a finite horizon. By this approach, the system of Hamilton-Jacobi-Bellman equations of a differential game derived from Bellman's optimality principle is transferred to a nonlinear system of algebraic equations solvable by using Newton's iteration method. Some illustrative examples are provided to show the accuracy and efficiency of the proposed numerical method.

Keywords. Differential games, Feedback Nash equilibrium, Bellman's optimality principle, Bernoulli Tau method.

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1. Introduction

In recent decades, considerable attention has been given to differential games due to their frequent appearance in various applications in science, engineering, economics and management [8, 11, 25, 33]. Differential game, as a combination of game theory, and optimal control theory, describes a conflict situation between several players seeking their own minimum costs under a dynamical system [7, 20, 31].

One of the most crucial and popular solution concepts in game theory is Nash equilibrium, whereby players have no incentive to deviate from their original plans [15, 18]. In feedback Nash equilibrium, the players are aware of the game running state at each time and choose their control strategies [13].

The system of Hamilton-Jacobi-Bellman (HJB) equations is derived from Bellman's optimality principle and plays a key role in the construction of feedback Nash equilibrium while it does not generally have an analytical solution [4]. Therefore, obtaining a numerical solution is at least the most logical approach to treat it.

Linear-quadratic (LQ) differential game is a branch of differential game problems and has arisen in many practical problems. In [30], the authors investigated the feedback Nash equilibrium of LQ differential games by considering their corresponding Riccati differential equations. The paper [17] proved the verification theorems of the dynamic programming type for determining the feedback strategies in differential games. A class of constrained LQ multistage games has been studied in [24] and its feedback Nash equilibrium has been characterized. An efficient computational method based on combining the policy iteration algorithm and Chebyshev spectral collocation method is presented in [19] for solving the stochastic LQ differential games. In [9], the authors considered a kind of stochastic LQ two-player differential game and discussed the existence of a closed-loop Nash strategy for it. In this paper, an appropriate numerical approach is presented for solving the system of HJB equations extracted from LQ differential games and computing their feedback Nash strategies.

Spectral methods are the most important approaches for the numerical solution of various kinds of ordinary and partial differential equations arising from different scientific and engineering problems with extreme accuracy. They are classified into three methods, namely Tau, Galerkin, and collocation. The basic mathematical theory of these has been established in many studies [6, 12, 29].

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The goal of the present study is to propose an efficient and accurate implementation of the Tau method for the system of HJB equations to approximate its solution functions as finite expansions of the Bernoulli polynomials and then compute the feedback Nash equilibrium in LQ differential games over a finite horizon.

The outline of this paper is as follows: A class of differential games is defined in the "Problem statement" section. In the "Bernoulli polynomials" section, the definition and properties of the Bernoulli polynomials and functions approximation by them are presented. A discussion on proposed numerical approach is in "The Bernoulli Tau method for solving differential games" section. An error analysis for the suggested approximation scheme is presented in the "Error estimation" section. In the "Numerical illustrations" section, several numerical problems are considered to demonstrate the accuracy and efficiency of the proposed method. A brief conclusion is presented in the "conclusion" section.

2. Problem statement

In this section, a class of differential game problems with the finite horizon is defined as follows:

Definition 2.1. A differential game over a finite planning horizon $[t_0, t_f]$ is described as the following [26]:

$$\begin{aligned}
&\min_{u_i\in U_i} J_i(u_i(\cdot), u_{-i}(\cdot)) = \int_{t_0}^{t_f} L_i(t, x(t), u_1(t), u_2(t), \cdots, u_n(t))\, dt \,+ \psi_i(t_f, x(t_f)),\\
&\dot{x}(t) = f(t, x(t), u_1(t), u_2(t), \cdots, u_n(t)), \qquad (2.1)\\
&x(t_0) = x_0 \in \mathbb{R},
\end{aligned}$$

where $x(t) \in \mathbb{R}$ is the state vector of this differential game and $u_i \in U_i \subset \mathbb{R}$, $i = 1, 2, \cdots, n$ describes the player i's control strategy, respectively. It is assumed that the functions $L_i(t, x(t), u_1(t), u_2(t) \ldots, u_n(t))$ and $\Psi_i(t_f, x(t_f))$, $i = 1, 2, \cdots, n$ representing the player i's running cost and terminal cost, respectively are continuously differentiable. The goal of solving this differential game is to minimize the players' costs by choosing the suitable control actions $u_i(.) \in U_i \subset \mathbb{R}$.

The following definition describes the feedback Nash equilibrium for differential game (2.1):

Definition 2.2 ([1])**.** The vector of control strategies $u^*(\cdot) = [u_i^*(\cdot)], i = 1, 2, \cdots, n$ constitutes a feedback Nash equilibrium for differential game (2.1) if for each $(t, x) \in \Omega := [t_0, t_f] \times [0, L]$ the following inequalities hold:

$$J_i(u_i^*(\cdot), u_{-i}^*(\cdot)) \le J_i(u_i(\cdot), u_{-i}^*(\cdot)), \quad \forall u_i \in U_i,$$

where u_i denotes player i's control strategy and u_{-i} state the other players' control strategies, i.e. $u_{-i} = u_j, j \neq i$.

To characterize a feedback Nash equilibrium for differential game (2.1), the value functions are defined as the following:

Definition 2.3 ([27])**.** For every single $(t, x) \in \Omega$ suppose

$$V_i(t, x) := J_i^*(t, x), \quad i = 1, 2, \ldots, n,$$

as the minimal obtainable costs in game (2.1), which starts at any random intermediate point at time t, with the assumption that the state of the game is at that specific time gained by x. Functions V_i, $i = 1, 2, \cdots, n$ are in a satisfactory relation with equations

$$V_i(t_f, x(t_f)) = \psi_i(t_f, x(t_f)), \quad i = 1, 2, \ldots, n.$$

These functions are labeled as the differential game's value functions (2.1).

Bellman's optimality principle gives a set of conditions for optimality of control strategies to build feedback Nash equilibrium in the differential game (2.1) like in the theorem below [2]:

Theorem 2.4. *Let $u_i^*(.)$, $i = 1, 2, \dots, n$ construct the feedback Nash equilibrium of differential game (2.1), and consider $x^*(t)$ as the corresponding closed-loop game state trajectory. Suppose that the value functions $V_i(.)$, $i = 1, 2, \dots, n$ are continuously differentiable. Then, these value functions satisfy the following system of first-order hyperbolic PDEs:*

$$\begin{cases} -\dfrac{\partial V_i}{\partial t} = \min\limits_{u_i \in U_i} \Big\{ L_i(t, x, u_1^*, u_2^*, \dots, u_{i-1}^*, u_i, u_{i+1}^*, \dots, u_n^*) + \dfrac{\partial V_i}{\partial x} f(t, x, u_1^*, u_2^*, \dots, u_{i-1}^*, u_i, u_{i+1}^*, \dots, u_n^*) \Big\}, \\ V_i(t_f, x(t_f)) = \psi_i(t_f, x(t_f)), \qquad\qquad i = 1, 2, \dots, n. \end{cases} \tag{2.2}$$

Proof. See [10]. □

The system of PDEs (2.2) is called the Hamilton-Jacobi-Bellman equation system that is a strongly nonlinear system in general making it very hard or almost impossible to solve in an analytical way. Thus, the application of a suitable numerical method is needed. When this system has been solved, the value functions V_i, $i = 1, 2, \cdots, n$, which were sought may be found, after which the feedback Nash equilibrium $(u_1^*(.), u_2^*(.), \dots, u_n^*(.))$ will be obtained, and the control strategies $u_i^*(.)$, $i = 1, 2, \dots, n$ will be gained as the following:

$$u_i^*(t, x) = arg \min_{u_i \in U_i} \left\{ L_i(t, x, u_1, u_2 \dots, u_n) + \frac{\partial V_i}{\partial x} f(t, x, u_1, u_2 \dots, u_n) \right\}.$$

Through the definition of the Hamiltonian function H_i, $i = 1, 2, \dots, n$ of player i, as in

$$H_i(t, x, u_1, u_2 \dots, u_n, \frac{\partial V_i}{\partial x}) = L_i(t, x, u_1, u_2 \dots, u_n) + \frac{\partial V_i}{\partial x} f(t, x, u_1, u_2 \dots, u_n),$$

and based on the Theorem 2.4, the determination procedure of the feedback Nash equilibrium in differential game (2.1) can be summarized as Algorithm 1.

Algorithm 1

Input: *The differential game (2.1).*

Step 1. *Write down the system of HJB equations; this system contains minimization operations.*

Step 2. *Minimize the Hamiltonian functions and find the optimal control strategies as $u_i^* = \Phi_i(t, x, \frac{\partial V_i}{\partial x})$, $i = 1, 2, \dots, n$.*

Step 3. *Insert the obtained optimal control strategies from Step 2 in the system of HJB equations in Step 1. This leads to a system in which no longer minimization operations appear, as the following:*

$$\begin{cases} -\dfrac{\partial V_i}{\partial t} = L_i(t, x, \Phi_1, \Phi_2, \cdots, \Phi_n) + \dfrac{\partial V_i}{\partial x} f(t, x, \Phi_1, \Phi_2, \cdots, \Phi_n), \\ V_i(t_f, x(t_f)) = \psi_i(t_f, x(t_f)), \quad i = 1, 2, \dots, n. \end{cases}$$

Step 4. *Solve the obtained system of HJB equations from Step 3 and find the value functions, namely $V_i(t, x)$, $i = 1, 2, \dots, n$.*

Step 5. *According to Steps 2 and 4, write down the optimal control strategies as the functions of the instant time t and the running state x.*

Output: *Feedback Nash equilibrium $(u_1^*(.), u_2^*(.), \dots, u_n^*(.))$.*

3. Bernoulli polynomials

In this study, Bernoulli polynomials are considered as basis functions for implementing the Tau method. The definition and properties of these polynomials in functions approximation are as the following:

Definition 3.1. Bernoulli polynomials of order n are defined on $[0,1]$ as [23]

$$\beta_n(t) = \sum_{i=0}^{n} \binom{n}{i} \alpha_{n-i} t^i,$$

where $\alpha_i, i = 0, 1, \dots, n$ are Bernoulli numbers and defined as

$$\frac{t}{e^t - 1} = \sum_{i=0}^{\infty} \alpha_i \frac{t^i}{i!}.$$

The first few Bernoulli numbers are

$$\alpha_0 = 1,\ \alpha_1 = \frac{-1}{2},\ \alpha_2 = \frac{1}{6},\ \alpha_4 = \frac{-1}{30},$$

with $\alpha_{2i+1} = 0$, for $i = 1, 2, 3, \dots$.

The first seven Bernoulli polynomials are [22]

$$\begin{aligned}
&\beta_0(t) = 1, \quad \beta_1(t) = t - \frac{1}{2}, \quad \beta_2(t) = t^2 - t + \frac{1}{6}, \quad \beta_3(t) = t^3 - \frac{3}{2}t^2 + \frac{1}{2}t,\\
&\beta_4(t) = t^4 - 2t^3 + t^2 - \frac{1}{30}, \quad \beta_5(t) = t^5 - \frac{5}{2}t^4 + \frac{5}{3}t^3 - \frac{1}{6}t,\\
&\beta_6(t) = t^6 - 3t^5 + \frac{5}{2}t^4 - \frac{1}{2}t^2 + \frac{1}{42}.
\end{aligned}$$

A complete basis is formed by these polynomials over the interval $[0,1]$. Therefore, any function $V(t) \in L^2(0,1)$ can be written as

$$V(t) = \sum_{j=0}^{\infty} v_j\, \beta_j^*(t),$$

where the coefficients v_j, $j = 1, 2, \dots,$ can be determined as [28]

$$v_j = \frac{1}{j!} \int_0^1 V^{(j)}(t) dt.$$

In practical applications, only the first $(M+1)$ terms of the Bernoulli polynomials are used for approximation purposes. Thus, the function $V(t)$ is written as

$$V(t) \simeq V^M(t) = \sum_{j=0}^{M} v_j \beta_j(t) = \nu^T B_M(t),$$

where $\nu^T = [v_0, v_1, \dots, v_M]$ is the vector of the expansion coefficients and

$$B_M(t) = [\beta_0(t), \beta_1(t), \dots, \beta_M(t)]^T \tag{3.1}$$

is the vector of the Bernoulli polynomials. In a similar way, an infinitely differentiable function $V(t,x)$ defined on $\Omega = [0,1] \times [0,1]$ can be approximated by a finite expansion of the double Bernoulli polynomials as

$$V(t,x) \simeq V^{MN}(t,x) = \sum_{j=0}^{M} \sum_{k=0}^{N} v^{jk}\, \beta_j(t)\beta_k(x) = B_M^T(t) W B_N(x),$$

where $B_M(t)$ and $B_N(x)$ are the Bernoulli vectors defining similarly to (3.1) and the expansion matrix W is given by

$$W = \begin{pmatrix} v^{00} & v^{01} & \cdots & v^{0N} \\ v^{10} & v^{11} & \cdots & v^{1N} \\ \vdots & \vdots & \cdots & \vdots \\ v^{M0} & v^{M1} & \cdots & v^{MN} \end{pmatrix},$$

where

$$v^{jk} = \frac{1}{j!k!}\int_0^1 \int_0^1 \frac{\partial^{j+k} V(t,x)}{\partial t^j \partial x^k} dtdx, \;\; j = 0,1,\dots,M, \;\; k = 0,1,\dots,N.$$

4. The Bernoulli Tau method for solving differential games

This section presents a methodical combining of the Tau method and the Bernoulli polynomials. The result of this combination is the Bernoulli Tau method (BTM), which is used for numerically solving the system of HJB equations (2.2), and gaining feedback Nash equilibrium for differential game (2.1). For simplification matters and without loss of generality [16], we consider $\Omega := [0,1]\times[0,1]$ and then, the unknown value functions $V_i(t,x)$, $i = 1,2,\cdots,n$ can be expanded by the double Bernoulli polynomials as

$$V_i(t,x) \simeq V_i^{MN}(t,x) = \sum_{j=0}^{M}\sum_{k=0}^{N} v_i^{jk}\,\beta_j(t)\beta_k(x) = B_M^T(t) W_i B_N(x), \tag{4.1}$$

where the matrices W_i, $i = 1,2,\cdots,n$ of $n(M+1)(N+1)$ unknown spectral coefficients are given by

$$W_i = \begin{pmatrix} v_i^{00} & v_i^{01} & \cdots & v_i^{0N} \\ v_i^{10} & v_i^{11} & \cdots & v_i^{1N} \\ \vdots & \vdots & \cdots & \vdots \\ v_i^{M0} & v_i^{M1} & \cdots & v_i^{MN} \end{pmatrix}.$$

The residual functions are defined by substituting these expansions in the partial differential equations of the system of HJB equations (2.2) as the following:

$$Res_i = \frac{\partial V_i}{\partial t} + L_i(t,x,\Phi_1,\Phi_2,\cdots,\Phi_n) + \frac{\partial V_i}{\partial x} f(t,x,\Phi_1,\Phi_2,\cdots,\Phi_n), \quad i = 1,\dots,n.$$

As in a typical Tau method [32], $nM(N+1)$ algebraic equations are generated in the unknown spectral coefficients, v_i^{jk}, $i = 1,2,\dots n$, $j = 0,1,\dots,M-1$, $k = 0,1,\dots,N$, namely

$$\int_0^1 \int_0^1 Res_i \beta_j(t)\beta_k(x) dtdx = 0. \tag{4.2}$$

The remaining of the required algebraic equations are extracted from the boundary conditions (2.2) as follows:

$$\Phi_M^T(1) W_i \Phi_N(x) = \psi_i(1,x), \quad i = 1,2,\dots n. \tag{4.3}$$

For collocating Equations (4.3) at $(N+1)$ points, the roots x_h, $h = 1,2,\dots,N+1$ of the shifted Legendre polynomial $P_{N+1}^*(x)$ on the interval $[0,1]$ are used as the suitable collocation points. Consequently, the unknown spectral coefficients can be obtained from Equations (4.2) and (4.3) and then $V_i^{MN}(t,x)$, $i = 1,2,\dots,n$ given in (4.1) are calculated.

5. Error estimation

In this section, upper bounds of the absolute errors between the examplect solutions $V_i(t,x),\ i=1,2,\dots,n$ and the approximate solutions $V_i^{MN}(t,x),\ i=1,2,\dots,n$ are provided by using the Lagrange interpolation polynomials. Consider

$$\mathbb{B}_{MN} = span\{\beta_j(t)\beta_k(x), j=0,1,\dots,M, \quad k=0,1,\dots,N\}.$$

Analytic expressions for the error norms of the best approximations of smooth functions $V_i(t,x)\in\Omega=[0,1]\times[0,1]$, $i=1,2,\dots,n$ by their expansions regarding the finite numbers of the double Bernoulli polynomials are presented according to the following theorem:

Theorem 5.1. *Let $V_i^{MN}(t,x)\in\mathbb{B}_{MN}$, $i=1,2,\dots,n$ be the best approximations of $V_i(t,x)$, $i=1,2,\dots,n$. Then*

$$\|V_i(t,x)-V_i^{MN}(t,x)\|_\infty \le \sigma_{1i}\frac{1}{2^{M+1}\omega_{M+1}(M+1)!}+\sigma_{2i}\frac{1}{2^{N+1}\omega_{N+1}(N+1)!}+\sigma_{3i}\frac{1}{2^{M+N+2}\omega_{M+1}\omega_{N+1}(M+1)!(N+1)!},$$

where

$$\max_{(t,x)\in\Omega}\left|\frac{\partial^{M+1}V_i(t,x)}{\partial t^{M+1}}\right|\le\sigma_{1i},\qquad \max_{(t,x)\in\Omega}\left|\frac{\partial^{N+1}V_i(t,x)}{\partial x^{N+1}}\right|\le\sigma_{2i},\qquad \max_{(t,x)\in\Omega}\left|\frac{\partial^{M+N+2}V_i(t,x)}{\partial t^{M+1}\partial x^{N+1}}\right|\le\sigma_{3i}.$$

Proof. By the definition of the best approximations for the value functions $V_i(t,x)$, $i=1,2,\dots,n$, it is concluded that the following inequalities hold:

$$\|V_i(t,x)-V_i^{MN}(t,x)\|_\infty \le \|V_i(t,x)-\nu_i^{MN}(t,x)\|_\infty,\quad \forall\ \nu_i^{MN}(t,x)\in\mathbb{B}_{MN}. \tag{5.1}$$

It is notable that the inequalities (5.1) also hold when $\nu_i^{MN}(t,x)$, $i=1,2,\dots,n$ are the interpolating polynomials for $V_i(t,x)$, $i=1,2,\dots,n$ at the points (t_j,x_k), where t_j, $j=0,1,\dots,M$ and x_k, $k=0,1,\dots,N$ are the roots of $P^*_{M+1}(t)$ and $P^*_{N+1}(x)$, respectively. Therefore, it results that [3, 14]

$$\begin{aligned} V_i(t,x)-\nu_i^{MN}(t,x) &= \frac{\partial^{M+1}V_i(\alpha,x)}{\partial t^{M+1}(M+1)!}\prod_{j=0}^{M}(t-t_j)+\frac{\partial^{N+1}V_i(t,\beta)}{\partial t^{N+1}(N+1)!}\prod_{k=0}^{N}(x-x_k)\\ &\quad -\frac{\partial^{M+N+2}V_i(\alpha',\beta')}{\partial t^{M+1}\partial x^{N+1}(M+1)!(N+1)!}\times\prod_{j=0}^{M}(t-t_j)\prod_{k=0}^{N}(x-x_k),\quad i=1,2,\dots,n, \end{aligned} \tag{5.2}$$

where $\alpha,\alpha'\in[0,1]$ and $\beta,\beta'\in[0,1]$. Taking the infinity norm from Equations (5.2), we obtain [21]

$$\begin{aligned} \|V_i(t,x)-\nu_i^{MN}(t,x)\|_\infty &\le \max_{(t,x)\in\Omega}\left|\frac{\partial^{M+1}V_i(\alpha,x)}{\partial t^{M+1}}\right|\frac{\left\|\prod_{j=0}^{M}(t-t_j)\right\|_\infty}{(M+1)!}\\ &+\max_{(t,x)\in\Omega}\left|\frac{\partial^{N+1}V_i(t,\beta)}{\partial t^{N+1}}\right|\frac{\left\|\prod_{k=0}^{N}(x-x_k)\right\|_\infty}{(N+1)!}\\ &+\max_{(t,x)\in\Omega}\left|\frac{\partial^{M+N+2}V_i(\alpha',\beta')}{\partial t^{M+1}\partial x^{N+1}}\right|\\ &\times\frac{\left\|\prod_{j=0}^{M}(t-t_j)\right\|_\infty\left\|\prod_{k=0}^{N}(x-x_k)\right\|_\infty}{(M+1)!(N+1)!},\ i=1,2,\dots,n. \end{aligned}$$

Since $V_i(t,x)$, $i=1,2,\dots,n$ are smooth functions on Ω, it is found that constants $\sigma_{1i},\sigma_{2i},\sigma_{3i}\in\mathbb{N}$, $i=1,2,\dots,n$ exist in such a way that

$$\begin{aligned}
&\max_{(t,x)\in\Omega}\Big|\frac{\partial^{M+1}V^i(t,x)}{\partial t^{M+1}}\Big|\le\sigma_{1i},\\
&\max_{(t,x)\in\Omega}\Big|\frac{\partial^{N+1}V^i(t,x)}{\partial x^{N+1}}\Big|\le\sigma_{2i},\\
&\max_{(t,x)\in\Omega}\Big|\frac{\partial^{M+N+2}V^i(t,x)}{\partial t^{M+1}\partial x^{N+1}}\Big|\le\sigma_{3i}.
\end{aligned}\tag{5.3}$$

The factor $\Big\|\prod_{j=0}^{M}(t-t_j)\Big\|_\infty$ is minimized as the following process. It is notable that the factor $\Big\|\prod_{k=0}^{N}(x-x_k)\Big\|_\infty$ can be minimized in a similar manner.
Using the one-to-one mapping $t=\frac{1}{2}(z+1)$ between the intervals $[-1,1]$ and $[0,1]$, it is concluded that

$$\begin{aligned}
\min_{t_j\in[0,1]}\max_{0\le t\le 1}\Big|\prod_{j=0}^{M}(t-t_j)\Big| &= \min_{z_j\in[-1,1]}\max_{-1\le z\le 1}\Big|\prod_{j=0}^{M}\frac{1}{2}(z-z_j)\Big|\\
&=(\frac{1}{2})^{M+1}\min_{z_j\in[-1,1]}\max_{-1\le z\le 1}\Big|\prod_{j=0}^{M}(z-z_j)\Big|\\
&=(\frac{1}{2})^{M+1}\min_{z_j\in[-1,1]}\max_{-1\le z\le 1}\Big|\frac{P_{M+1}(z)}{\omega_{M+1}}\Big|,
\end{aligned}\tag{5.4}$$

where $\omega_{M+1}=\frac{(2M+2)!}{2^{M+1}((M+1)!)^2}$ and z_j, $j=0,1,\dots,M$ are the leading coefficient and the roots of $P_{M+1}(z)$, respectively. From this fact that

$$\max_{-1\le z\le 1}\Big|P_{M+1}(z)\Big|=P_{M+1}(1)=1,$$

and together with (5.3) and (5.4), the desired result is obtained as the following:

$$\|V_i(t,x)-V_i^{MN}(t,x)\|_\infty\le\sigma_{1i}\frac{1}{2^{M+1}\omega_{M+1}(M+1)!}+\sigma_{2i}\frac{1}{2^{N+1}\omega_{N+1}(N+1)!}+\sigma_{3i}\frac{1}{2^{M+N+2}\omega_{M+1}\omega_{N+1}(M+1)!(N+1)!},$$

□

6. Numerical illustrations

In this section, two differential game problems are presented to verify the accuracy and efficiency of the present numerical approach. The analytical solution of Example 6.1 is available and therefore, the applicability of the proposed approach is validated by being compared with the exact solution. Example 6.2 is a kind of LQ differential game with no exact solution. To investigate the effectiveness of the suggested approach for this problem, the norm of residuals error is considered. By these examples, it is observed that the proposed numerical approach is a convergent method with highly accurate results, which also has this advantage that the number of the required Bernoulli polynomials is small.

example 6.1. For this LQ differential game, the state equation is [10]

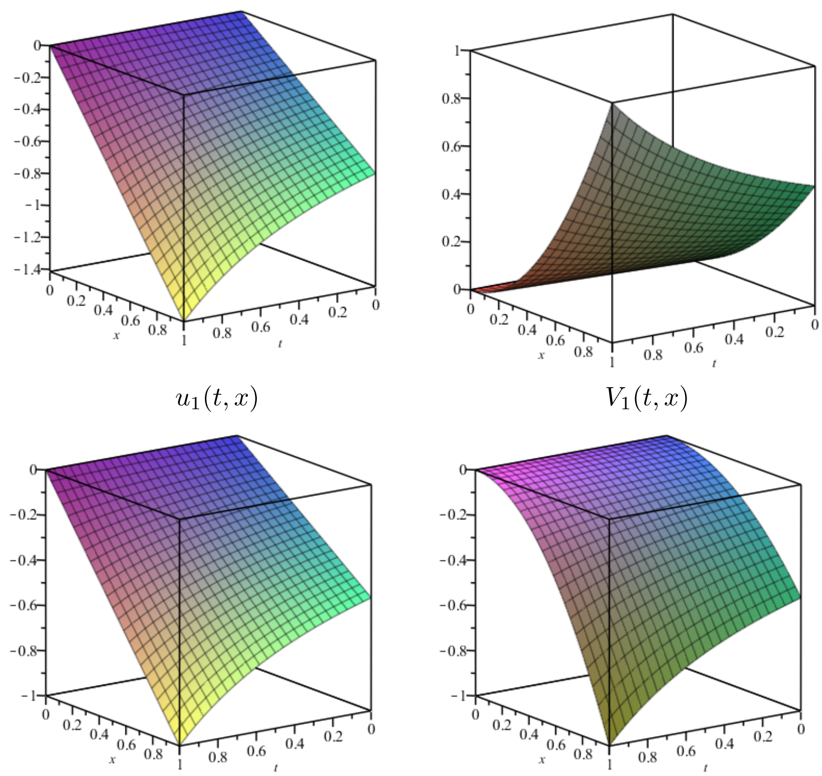
$$\dot{x}(t)=\sqrt{2}u_1(t)-u_2(t),\quad x(0)=1,$$

and two players' performance indices are as

$$\min_{u_1}J_1=\int_0^1(u_1^2(t)-u_2^2(t))dt+x^2(1),$$

and

$$\min_{u_2} J_2 = \int_0^1 (-u_1^2(t) + u_2^2(t))dt - x^2(1).$$



$u_1(t,x)$ $V_1(t,x)$

$u_2(t,x)$ $V_2(t,x)$

FIGURE 1. Plots of the approximate solutions with $M = 5$ and $N = 2$ for example 6.1

The examplect values of players' performance indices for this feedback Nash LQ differential game are [10]

$$J_1^* = 0.5,\ J_2^* = -0.5.$$

The nonlinear system of HJB equations derived from Bellman's optimality principle for this differential game is stated as follows:

$$\begin{cases} \frac{\partial V_1}{\partial t} + 0.5(\frac{\partial V_1}{\partial x})^2 + 0.25(\frac{\partial V_2}{\partial x})^2 + 0.5\frac{\partial V_1}{\partial x}\frac{\partial V_2}{\partial x} = 0, \\ \frac{\partial V_2}{\partial t} + 0.5(\frac{\partial V_1}{\partial x})^2 + 0.25(\frac{\partial V_2}{\partial x})^2 + \frac{\partial V_1}{\partial x}\frac{\partial V_2}{\partial x} = 0, \\ V_1(1,x) = x^2(1), \quad V_2(1,x) = -x^2(1). \end{cases}$$

The numerical results of the performance indices governed by the present method for different values of M and fixed $N = 2$ on $\Omega = [0,1] \times [0,1]$ and the absolute errors are shown in Table 1. From Table 1, it is observed that as M increases the absolute errors are reduced. Furthermore, the approximate solutions of $V_1(t,x)$, $V_2(t,x)$, $u_1(t,x)$, and $u_2(t,x)$ with $M = 5$ and $N = 2$ are plotted in Figure 1.

TABLE 1. Optimal payoff functionals J_1 and J_2 obtained by BTM for various values of M and fixed $N = 2$ on $\Omega = [0, 1] \times [0, 1]$ and the absolute errors for example 6.1.

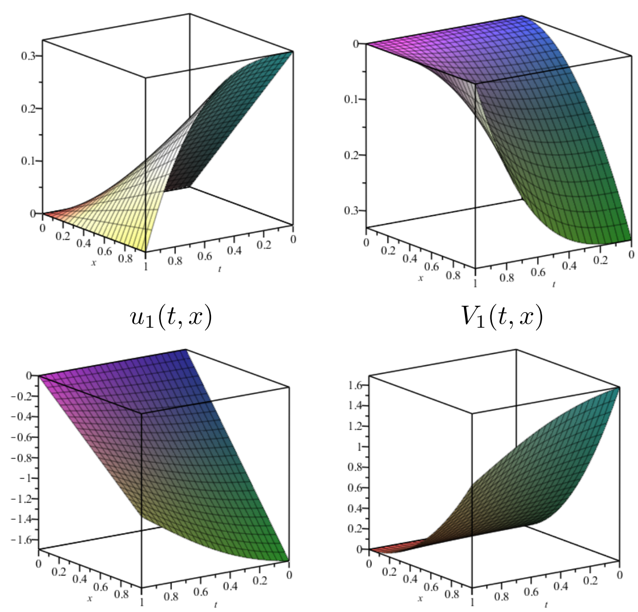
M	J_{1BTM}	J_{2BTM}	$\lvert J_{iBTM} - J_i^*\rvert, i = 1, 2$
3	0.499998998476	−0.499998998476	10^{-6}
4	0.499999973741	−0.499999973741	2.62×10^{-8}
5	0.499999999275	−0.499999999275	7.24×10^{-10}

example 6.2. For this LQ differential game, the state equation is [10]

$$\dot{x}(t) = u_1(t) + u_2(t), \quad x(0) = 1,$$

and two players' performance indices are as

$$\min_{u_1} J_1 = \int_0^1 (-x^2(t) + u_1^2(t))dt,$$
$$\min_{u_2} J_2 = \int_0^1 (2x^2(t) + u_2^2(t))dt + x^2(1).$$



$u_1(t,x)$ $V_1(t,x)$ $u_2(t,x)$ $V_2(t,x)$

FIGURE 2. Plots of the approximate solutions with $M = 8$ and $N = 2$ for example 6.2

The nonlinear system of HJB equations derived from Bellman's optimality principle for the differential game under consideration is stated as the following:

$$\begin{cases} \dfrac{\partial V_1}{\partial t} - x^2 - 0.25(\dfrac{\partial V_1}{\partial x})^2 - 0.5\dfrac{\partial V_1}{\partial x}\dfrac{\partial V_2}{\partial x} = 0, \\ \dfrac{\partial V_2}{\partial t} + 2x^2 - 0.25(\dfrac{\partial V_2}{\partial x})^2 - 0.5\dfrac{\partial V_1}{\partial x}\dfrac{\partial V_2}{\partial x} = 0, \\ V_1(1, x) = 0, \quad V_2(1, x) = x^2(1). \end{cases}$$

The numerical approximations of the performance indices governed by the present method for various values of M and fixed $N = 2$ on $\Omega = [0,1] \times [0,1]$ are presented in Table 2. Also, the approximate solutions of $V_1(t,x)$, $V_2(t,x)$, $u_1(t,x)$ and $u_2(t,x)$ with $M = 8$ and $N = 2$ are plotted in Figure 2. It is worth mentioning that since the examplect solution of this differential game is not available, the norm of residuals error is defined as follows to check the accuracy and validity of the proposed method for the differential game under consideration:

$$\|Res\|^2 = \int_0^1 \int_0^1 (Res_1^2 + Res_2^2) dtdx.$$

It is seen from Table 2 that as N increases, the norm of residuals error is reduced.

Table 2. Optimal payoff functionals J_1 and J_2 obtained by BTM for various values of M and fixed $N = 2$ on $\Omega = [0,1] \times [0,1]$ with the norm of residuals error for example 6.2.

N	J_{1BTM}	J_{2BTM}	$\|Res\|^2$
4	−0.33012918	1.69257708	1.22×10^{-6}
6	−0.33012777	1.69260413	7.28×10^{-9}
8	−0.33012776	1.69260408	7.97×10^{-12}

7. Conclusions

In this paper, an efficient combination of the Tau method with the Bernoulli polynomials known as the Bernoulli Tau method (BTM) was established effectively to compute the feedback Nash equilibrium in differential games with the finite horizon. By this method, the system of HJB equations derived from Bellman's optimality principle was reduced to a system of nonlinear algebraic equations solvable by using Newton's iteration method to consequently find the feedback Nash equilibrium. Moreover, the error estimation of the present approximation scheme was carried out by a theorem. Finally, two examples of different kinds of differential games were presented and solved to demonstrate the accuracy and efficiency of the proposed approach. It was observed that only a small number of the Bernoulli polynomials are needed to obtain satisfactory results.

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