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Asymptotic distributions of Neumann problem for Sturm-Liouville equation

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Abstract In this paper we apply the Homotopy perturbation method to derive the higher-order asymptotic distribution of the eigenvalues and eigenfunctions associated with the linear real second order equation of Sturm-liouville type on $[0, \pi]$ with Neumann conditions $(y'(0) = y'(\pi) = 0)$ where q is a real-valued Sign-indefinite number of $C^1[0, \pi]$ and λ is a real parameter.

Keywords. Sturm-Liouville, Nondefinite problem, Homotopy perturbation method, Asymptotic distribution.

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1. Introduction

The theory of the boundary eigenvalue problems associated with Sturm-liouville equation of the form

$$y'' + (\lambda r(x) - q(x))y = 0, \tag{1.1}$$

where q(x) is assumed continuous dates back to the pioneering research of R.G.D Richardson and O.Haupt(see [9] and the reference therein for a brief history and survey). The leading term in the asymptotic expansion of the real eigenvalue was the subject of the Jorgen's conjecture dating from 1964, a conjecture that was finally proved and extended in [1]. The thrust of this conjecture is that, once suitably related, the positive λ_m^+(resp. negative λ_m^-) eigenvalues admit the asymptotic estimate $\lambda_m^\pm \sim \frac{\pm m^2\pi^2}{(\int_a^b \sqrt{r_\pm(x)dx})^2}$ as $m \longrightarrow \infty$ where $r_\pm(x)$ represent the part of the weight function r, thus, $r_\pm(x) = max(\pm r(x), 0)$. Higher order asymptotic distribution of the eigenvalues was obtained by different methods. For example Liouville transforms (1.1) by setting

$$\xi(x_0) = \int \sqrt{r(x)}dx, \;\; y(x) = r^{-\frac{1}{4}}(x)W(\xi),$$

and obtained

$$\frac{d^2W}{d\xi^2} + (\lambda - R(x))W = 0, \tag{1.2}$$

where

$$R(\xi) = \frac{r''(x)}{4r^2(x)} - \frac{5r'^2}{16r^3(x)} + \frac{q(x)}{r(x)} \tag{1.3}$$

$$= -\frac{d^2}{r^{\frac{3}{4}}dx^2}(\frac{1}{r^{\frac{1}{4}}}) + \frac{q(x)}{r(x)}. \tag{1.4}$$

He used the method of variation of parameters to show that the solutions of the equation (1.2) satisfy in Volterra integral equation

$$y(x) = c_1\cos(\sqrt{\lambda}x) + c_2\sin(\sqrt{\lambda}x) + \frac{1}{\sqrt{\lambda}}\int_a^x \sin\sqrt{\lambda}(x-t)R(t)y(t)dt,$$

and proposes to solve this equation by the method of successive approximations where c_1 and c_2 are some constants. One can apply the above Volterra integral equation to obtain the asymptotic expressions for the eigenvalues of the equation (1.2) on $[0,\pi]$ with the boundary conditions $y(0) = y(\pi) = 0$ of the form

$$\lambda_n = n + 1 + \frac{1}{2n\pi}\int_0^\pi q(x)dx + o(\frac{1}{n^2}),$$

where the corresponding eigenfunction of the eigenvalue λ_n vanishes n times within the interval $(0,\pi)$. More details can be found in [6, 7, 8, 9, 12]. Hochstad assumed that the equation (1.2) has a solution of the form

$$y(x) = A(x)\sin w(x), \qquad y'(x) = \sqrt{\lambda - R(x)}A(x)\cos w(x),$$

by deriving a new differential equation, one can determine $A(x)$ and $w(x)$. By assuming that $R(x)$ has mean value 0, he found the fifth approximation for the eigenvalues of the equation (1.2) with the boundary conditions $y(0) = y(\pi) = 0$. It is of the form

$$\sqrt{\lambda_n} = n + \frac{\int_0^\pi q(x)^2dx - q'(\pi) + q'(0)}{8n^3\pi} + O(\frac{1}{n^4}).$$

For more details see ([5], P.154). In [2] the asymptotic formula for the eigenvalues and eigenfunctions of Sturm-Liouville problem with the Dirichlet boundary conditions are obtained by using homotopy perturbation method. In this paper we consider the equation (1.1), when $r(x) = 1$, with Neumann conditions ($y'(0) = y'(\pi) = 0$) and continue the study of the eigenvalues and eigenfunctions of this problem and exhibit higher order terms in the asymptotic expansions by using HPM.

2. Homotopy perturbation method

Recently a great deal of interest has been focused on the application of homotopy perturbation method, because this method is to continuously deform a simple problem, easy to solve, in to the difficult problem under study. The HPM has bean applied with great success to obtain the approximate solution of large variety of linear and nonlinear problems in ordinary differential equations, partial differential equations and integral differential equations. The homotopy theory becomes a powerful mathematical tool, when it is successfully coupled with perturbation theory [2, 3, 4, 10, 11, 13]. To describe HPM, we consider the following equation

$$L(\nu) + N(\nu) - f(r) = 0, \tag{2.1}$$

where L is linear and N is nonlinear operator. By the homotopy technique, a homotopy on $R \times [0,1]$ which satisfies

$$H(\nu,p) = (1-p)[L(\nu) - L(u_0)] + p[L(\nu) + N(\nu) - f(r)], \tag{2.2}$$

or

$$H(\nu,p) = H(\nu) - (1-p)L(u_0) + p[N(\nu) - f(r)] = 0, \tag{2.3}$$

is constructed, where $p \in [0,1]$ is an imbedding parameter, ν_0 is an initial approximation of $L(\nu) = 0$. Hence

$$H(\nu,0) = L(\nu) - L(\nu_0), \qquad H(\nu,1) = L(\nu) + N(\nu) - f(r),$$

and the changing process of p from 0 to 1, is just that of $H(\nu,p)$ from $L(\nu) - L(\nu_0)$ to $L(\nu) + N(\nu) - f(r)$. In topology, this is called deformation, $L(\nu) - L(\nu_0)$ and $L(\nu) + N(\nu) - f(r)$ are called homotopic. Applying the perturbation technique, due to the fact that $0 \le p \le 1$ can be considered as a small parameter, we can assume that the solution of equation (2.1) can be written as a series in p

$$\nu = \nu_0 + p\nu_0 + p^2\nu_2 + p^3\nu_3 + ..., \tag{2.4}$$

when $p \to 1$, (3.2) becomes the approximate solution of (2.1).

3. Results

In this section we apply HPM to derive higher-order asymptotic distribution of eigenvalues and eigenfunctions of Sturm-liouville problem. Consider the equation

$$y'' + \lambda y(x) = q(x)y, \qquad 0 \le x \le \pi, \tag{3.1}$$

with the Neumann conditions ($y'(0) = y'(\pi) = 0$) where $q(x)$ is a real valued square integrable function on $[0,\pi]$, i.e., $q(x) \in L_2[0,\pi]$. Now we construct the homotopy $\nu(x,p) : [0,\pi] \times [0,1] \to \ R$ with

$$H(\nu,p) = (1-p)[L(\nu) - L(\nu_0)] + p[L(\nu) - q(x)\nu] = 0, \tag{3.2}$$

where $p \in [0,1]$ is an embedding parameter, $\nu_0 = A\cos(nx)$ is the initial approximation which satisfies the boundary conditions and $L = \frac{d^2}{dx^2} + \lambda$ is the auxiliary linear operator. If $p = 0$ then

$$H(\nu,0) = L(\nu) - L(\nu_0) = 0,$$

and if $p = 1$ then

$$H(\nu,1) = L(\nu) - q(x)\nu = 0.$$

Therefore, as the embedding parameter p increases from 0 to 1, the solution of the equation

$$H(\nu,p) = 0,$$

depends upon the embedding parameter p and varies from the initial approximation $\nu_0(x)$ to the solution $y(x)$ of equation (3.1). In topology, such a kind of continuous variation is called deformation. Assume that the solution ν of (3.2) and the eigenvalue λ can be written as a power series in p

$$\nu = \nu_0 + p\nu_1 + p^2\nu_2 + ..., \tag{3.3}$$

$$\lambda = \lambda_0 + p\lambda_1 + p^2\lambda_2 + ..., \tag{3.4}$$

and then the approximation solutions are obtained by

$$u = \lim_{p\to 1} \nu = \nu_0 + \nu_1 + \nu_2 + ...,$$

$$\Lambda = \lim_{p\to 1} \lambda = \lambda_0 + \lambda_1 + \lambda_2 + ...$$

Substituting (3.3) in to (3.2), and equating coefficients of p, we have the following equations

$$\nu_0'' + \lambda_0\nu_0 = 0,\ \nu_0'(0) = \nu_0'(\pi) = 0, \tag{3.5}$$

$$\nu_1'' + \lambda_0\nu_1 = -\lambda_1\nu_0 + q(x)\nu_0,\ \nu_1'(0) = \nu_1'(\pi) = 0, \tag{3.6}$$

$$\nu_2'' + \lambda_0\nu_2 = -\lambda_2\nu_0 - \lambda_1\nu_1 + q(x)\nu_1,\ \nu_2'(0) = \nu_2'(\pi) = 0, \tag{3.7}$$

$$\vdots$$

or

$$L(\nu_k) = -\sum_{i=0}^{k-1} \lambda_{k-i}\nu_i + q(x)\nu_{k-1},\ \nu_k'(0) = \nu_k'(\pi) = 0,\ k \geq 1, \tag{3.8}$$

where $\nu_0 = A\cos(mx)$, $m = 1, 2, 3, ...$ Using the boundary conditions $y'(0) = y'(\pi) = 0$ and since the set $\{\sqrt{2}\cos(jx)\ \ : j = 1, 2, 3, ...\}$ forms an orthonormal basis for $L_2[0, \pi]$, it is straightforward that the eigenfunction $\nu_k^{(n)}$ can be expressed by the set of base functions

$$\{\cos(jx)|j = 1, 2, 3, ...\}$$

in the form

$$\nu_k^{(n)} = \sum_{j=1}^{\infty} c_{kj}\cos(jx), \tag{3.9}$$

where $c_{kj} = \langle \nu_k^{(n)}, \cos(jx)\rangle$ and $\langle f, g\rangle = \int_0^\pi f(x)g(x)dx$. This provides us with the so-called rule of solution expression. Substituting (3.9) into (3.6) and simplifying it, we obtain, due to considering the shifting assumption $\int_0^1 q(x)dx = 0$, without loss of generality,

$$\sum_{j=1}^{\infty}(\lambda_0^{(n)} - \lambda_0^{(j)})c_{1j}\cos(jx) + A(\lambda_1 - q(x))\cos(nx) = 0.$$

Multiplying this equation by $\cos(mx)$ and then integrating the result over$[0, \pi]$ yields

$$\sum_{j=1}^{\infty}(\lambda_0^{(n)} - \lambda_0^{(j)})c_{1j}\int_0^1 \cos(jx)\cos(mx)dx +$$

$$+A\int_0^1 (\lambda_1 - q(x))\cos(nx)\cos(mx)dx = 0.$$

Solving the above equation, we have

$$\lambda_1 = 2\langle q(x)\cos(nx), \cos(nx)\rangle, \tag{3.10}$$

for the case $m = n$. Also, for the case $m \neq n$ we deduce

$$c_{1m}(\lambda_0^{(n)} - \lambda_0^{m})\|\cos(nx)\|^2 - A\langle q(x)\cos(nx), \cos(mx)\rangle = 0.$$

So

$$\lambda_1 = \langle \cos(2nx), q(x)\rangle, \tag{3.11}$$

$$c_{1m} = A\frac{\langle q(x), \cos(n+m)x\rangle + \langle q(x), \cos(n-m)x\rangle}{n^2 - m^2} \qquad m \neq n. \tag{3.12}$$

Therefore, if we set $q_j = \langle q(x)\cos(jx)\rangle$ the solution of (3.6) is

$$\nu_1^{(n)} = c_{1n}\cos(nx) + A\sum_{m\neq n;m=1}^{\infty}\frac{(q_{n+m} + q_{n-m})}{n^2 - m^2}\cos(mx). \tag{3.13}$$

Hence the second order approximation of the eigenfunctions are obtained

$$\nu_0^{(n)} + \nu_1^{(n)} = (A + c_{1n})\cos(nx) + \sum_{m\neq n;m=1}^{\infty} c_{1m}\cos(mx).$$

From the normalized condition we have $A + c_{1n} = \sqrt{2}$ and so we obtain

$$\nu_0^{(n)} + \nu_1^{(n)} = \sqrt{2}\cos(nx) + (\sqrt{2} - c_{1n})\sum_{m\neq n;m=1}^{\infty}\frac{(q_{n+m} + q_{n-m})}{n^2 - m^2}\cos(mx).$$

Because of the assumption $\frac{d^2}{dx^2}\nu_k^{(n)} \in L_2[0,\pi]$ it follows $\sum_{j=1}^{\infty}|c''_{kj}|^2 < \infty$ and since $c_{kj} = -\frac{1}{j^2}c''_{kj}$ then $\sum_{j=1}^{\infty} j^4|c_{kj}|^2 < \infty$. Hence $c_{kn} = O(\frac{1}{n^{\delta+5/2}})$ for $\delta > 0, k \geq 1$. From

$$\sum_{m\neq n;m=1}^{\infty}\frac{1}{n^2 - m^2} = O(\frac{\ln(n)}{n}),$$

We deduce

$$\nu_0^{(n)} + \nu_1^{(n)} = \sqrt{2}\cos(nx) + \sqrt{2}\sum_{m\neq n;m=1}^{\infty}\frac{(q_{n+m} + q_{n-m})}{n^2 - m^2}\cos(mx) + \\ + O(\frac{\ln(n)}{n^{\delta+7/2}}).$$

Similarly, substituting (3.3) into (3.6), simplifying it, multiplying the result by $\cos(mx)$ and then integrating over$[0,\pi]$ yields

$$-\frac{A}{2} - \frac{1}{2}\lambda_1 c_{1n} + c_{1n}\langle q(x)\cos(nx), \cos(nx)\rangle + \\ + \sum_{m\neq n;m=1}^{\infty} c_{1m}\langle q(x)\cos(mx), \cos(nx)\rangle = 0,$$

for the case $m = n$. Therefore

$$\lambda_2 = \sum_{m_1\neq n;m_1=1}^{\infty}\frac{(q_{n+m_1} + q_{n-m_1})^2}{n^2 - m_1^2}. \tag{3.14}$$

Also, for the case $m \neq n$ we obtain

$$\frac{1}{2}c_{2m}(n^2-m^2) = c_{1n}\langle q(x)\cos(nx), \cos(mx)\rangle + A\sum_{m_1\neq n; m_1=1}^{\infty}\frac{(q_{n+m_1}+q_{n-m_1})}{n^2-m_1^2}$$

$$\times(-\lambda_1\langle\cos(m_1x), \cos(mx)\rangle + \langle q(x)\cos(m_1x), \cos(mx)\rangle).$$

So

$$c_{2m} = c_{1n}\frac{(q_{n+m}+q_{n-m})}{(n^2-m^2)} - A\frac{q_{2n}(q_{n+m}+q_{n-m})}{(n^2-m^2)^2} +$$

$$+A\sum_{m_1\neq n; m_1=1}^{\infty}\frac{(q_{n+m_1}+q_{n-m_1})(q_{m_1+m}+q_{m_1-m})}{(n^2-m^2)(n^2-m_1^2)}.$$

Thus the three-order approximations for eigenvalues is

$$\lambda_0^{(n)} + \lambda_1^{(n)} + \lambda_2^{(n)} = n^2 + q_{2n} + \sum_{m_1\neq n; m_1=1}^{\infty}\frac{(q_{n+m_1}+q_{n-m_1})^2}{n^2-m_1^2},$$

and for eigenfunctions

$$\nu_0^{(n)} + \nu_1^{(n)} + \nu_2^{(n)} = (A + c_{1n} + c_{2n})\cos(nx) +$$

$$(A+c_{1n})\sum_{m_1\neq n; m\geq 1}\frac{(q_{n+m_1}+q_{n-m_1})^2}{n^2-m_1^2}\cos(m_1x)$$

$$-A\sum_{m_1\neq n; m\geq 1}\frac{q_{2n}(q_{n+m_1}+q_{n-m_1})}{(n^2-m_1^2)^2}\cos(m_1x)$$

$$+A\sum_{m_1,m_2\geq 1; m_1,m_2\neq n}\frac{(q_{n+m_1}+q_{n-m_1})(q_{m_1+m_2}+q_{m_1-m_2})}{(n^2-m_1^2)(n^2-m_2)}\cos(m_2x).$$

From the normalized condition we must have $A + c_{1n} + c_{2n} = \sqrt{2}$ and similar to previous approximation we obtain

$$\nu_0^{(n)} + \nu_1^{(n)} + \nu_2^{(n)} = \sqrt{2}\cos(nx) + \sqrt{2}\sum_{m_1\neq n; m_1\geq 1}\frac{(q_{n+m_1}+q_{n-m_1})}{n^2-m_1^2}\cos(m_1x)$$

$$-\sqrt{2}\sum_{m_1\neq n; m_1\geq 1}\frac{q_{2n}(q_{n+m_1}+q_{n-m_1})}{(n^2-m_1^2)^2}\cos(m_1x)$$

$$+\sqrt{2}\sum_{m_1,m_2\geq 1; m_1,m_2\neq n}\frac{(q_{n+m_1}+q_{n-m_1})(q_{m_1+m_2}+q_{m_1-m_2})}{(n^2-m_1^2)(n^2-m_2)}\cos(m_2x)$$

$$+O(\frac{\ln(n)}{n^{\delta+7/2}}),$$

where the first, second and the last sums are of order $O(\frac{\ln n}{n})$, $O(\frac{\ln n}{n^3})$and $O((\frac{\ln n}{n})^2)$ respectively.

References

[1] F. V. Atkinson, A. B. Mingarelli, Asymptotics of the number of zeros and the eigenvalues of general weighted Sturm-Liouville problems, *J. Reine angev. Math. 375* (1987), 380-393.

[2] M. Duman, Asymptotics for the Sturm-Liouville problem by homotopy perturbation method, *Applied Mathematics and Computation, 216* (2010), 492-496.

[3] J. H. He, Homotopy perturbation technique, *computer Methods in Applied mathematics and Engineering, 178* (1999), 257-262.

[4] J. H. He, Homotopy perturbation method: a new nonlinear technique, *Applied Mathematics and Computation, 135* (2003), 73-79.

[5] H. Hochstsdt, Differential equations, *Dover,* New york, 1957.

[6] E. L. Ince, Ordinary differential equations, *Dover,* New york,1956.

[7] B. M. Levitan, G. Gasymov, Determination of a differential equation by its spectra, *Russian Math. Surveys. 19* (1964), 1-63.

[8] B. M. Levitan, G. Gasymov, Introduction to spectral theory, *Translation of Math. Amer. Math. Society* (1975).

[9] A. B. Mingarelli, A survey of the regular weighted Sturm-Liouville problem, The non-definite case, *P. Fuquan, X. Shutie(Eds.), Applied Differential Equations, World scientific,* Singapoure and Philadelphia(1986), 109-137.

[10] Z. M. Odibat, A new modification of the homotopy perturbation method for linear and nonlinear operators, *Applied Mathematics and Computation, 189* (2007), 746-753.

[11] J. I. Ramos, Piecewise homotopy methods for nonlinear ordinary differential equations, *Appled Mathematics and Computation, 198* (2008), 92-116.

[12] F. G. Tricomi, Differential equations, *Hofner,* New york, 1961.

[13] E. yusufoglu, A homotopy perturbation algorithm to solve a system of Fredholm-Volterra type integral equations, *Mathematical and Computer Modeling, 47* (2008), 1099-1107.