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European option pricing of fractional Black-Scholes model with new Lagrange multipliers

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Abstract In this paper, a new identification of the Lagrange multipliers by means of the Sumudu transform, is employed to obtain a quick and accurate solution to the fractional Black-Scholes equation with the initial condition for a European option pricing problem. Undoubtedly this model is the most well known model for pricing financial derivatives. The fractional derivatives is described in Caputo sense. This method finds the analytical solution without any discretization or additive assumption. The analytical method has been applied in the form of convergent power series with easily computable components. Some illustrative examples are presented to explain the efficiency and simplicity of the proposed method.

Keywords. Sumudu transforms, Fractional Black- Scholes equation, European option pricing problem, Lagrange multipliers.

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1. Introduction

A financial derivative is an instrument whose price depends on, or is derived from, the value of another asset [15]. Often, this underlying asset is a stock. The concept of financial derivatives is not new. While there remains some historical debate as to the exact date of the creation of financial derivatives, it is well accepted that the first attempt at modern derivative pricing began with the work of Charles Castelly [5] published in 1877. Castelly's book was a general introduction to concepts such as hedging and speculative trading, but it laked mathematical rigor. In 1969, Fisher Black and Myron Scholes got an idea that would change the world of finance forever. The central idea of their paper revolved around the discovery that one did not need to estimate the expected return of a stock in order to price an option written on that stock. The Black-Scholes model (BS) for pricing stock options has been applied to many different commodities and payoff structures. The Black-Scholes model for value of an option is described by the following equation:

$$\frac{\partial v}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 v}{\partial x^2} + r(t) x \frac{\partial v}{\partial x} - r(t) v = 0, \quad (x,t) \in R^+ \times (0,T), \tag{1.1}$$

where $v(x,t)$ is the European option price at asset price x and at time t, T is the maturity, $r(t)$ is the risk free interest rate and $\sigma(x,t)$ represents the volatility function of underlying asset. Let us denote by $c(x,t)$ and $p(x,t)$ the value of the European call and put options, respectively. Then, the payoff functions are

$$c(x,t) = \max(x-E,0) \quad , \quad p(x,t) = \max(E-x,0),$$

where E is the exercise price. The classical Black-Scholes equation was established under some strict assumptions. Therefore, some improved models have been proposed to weaken these assumptions, such as models with transactions costs [8, 2], stochastic volatility model [16], Jump-diffusion model [24], and stochastic interest model [23]. With the discovery of the fractal structure for financial market, the fractional Black-Scholes models [4, 27, 20] are derived by replacing the standard Brownian motion involved in the classical model with fractional Brownian motion. Since the fractional Brownian motion is not a semi-martingale, the arbitrage opportunities exist in the fractional Black-Scholes model under a complete and frictionless setting.
Fractional differential operators have a long history, having been mentioned by Leibniz in a letter to de L'Hopital in 1695. However, it is in the past hundred years that the most intriguing leaps in engineering and scientific application have been found. It is widely and efficiently used to describe many phenomena arising in engineering, physics, economy, and science. A family of numerical [9, 21, 11], semi-analytical [10, 14], and analytical methods has been developed for solving ordinary and fractional differential equations [18, 22].
The variational iteration method, first proposed by He [14], is a modified general Lagranges multiplier method [17]. This method is a modification of the general Lagrange multiplier method into an iteration method, which is called correction functional. The major problem of the variational iteration method is the correct determination of the Lagrange multiplier, when the method is applied to ordinary and fractional equations. It is difficult for one to use the integration by parts to derive the Lagrange multipliers explicitly. In this work, a new modification of variational iteration method is considered, which is based on the Sumudu transform. Therefore, we apply a new Lagrange multiplier for pricing European option of fractional version of the Black-Scholes model.

2. Preliminaries

Definition 2.1. A real function $y(t)$, $t>0$, is said to be in the space C_μ, $\mu \in \mathbb{R}$ if there exists a real number $p(>\mu)$, such that $y(t)=t^p y_1(t)$, where $y_1(t) \in C[0,\infty]$, and it is said to be in the space C_μ^m iff $y^{(m)} \in C_\mu, m \in \mathbb{N}$.
The Riemann-Liouville fractional integral and Caputo derivative are defined as follows.
Definition 2.2. The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$, of a function
$y \in C_\mu,\ \mu \geq -1$, is defined as:

$$J_t^\alpha y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} y(\tau) d\tau,\ \alpha>0,\ t>0,\ J^0 y(t) = y(t).$$

Some of the most important properties of operator J^α for $y \in C_\mu,\ \mu \geq -1,\ \alpha,\ \beta \geq 0$ and $\gamma > -1$, are as follows [26]:
1. $J^\alpha J^\beta y(t) = J^{(\alpha+\beta)} y(t)$;
2. $J^\alpha J^\beta y(t) = J^\beta J^\alpha y(t)$;
3. $J^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} t^{\alpha+\gamma}$.

Definition 2.3. The fractional derivative of $y(t)$ in the Caputo sense is defined as:

$$D_t^\alpha y(t) = J_t^{m-\alpha} D_t^m y(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} y^{(m)}(\tau) d\tau,$$

for $m-1 < \alpha \leq m,\ m \in \mathbb{N},\ t > 0,\ y \in C_{-1}^m$.
Note that the relation between Riemann-Liouville fractional integral operator and modified Riemann-Liouville fractional differential operator is given by Fractional Leibnitz formulation as follows:

$$J_t^\alpha D_t^\alpha y(t) = D_t^{-\alpha} D_t^\alpha y(t) = y(t) - \sum_{k=0}^{m-1} \frac{t^k}{k!} y^k(0), \quad m-1 < \alpha \leq m.$$

Definition 2.4. The Sumudu transform is defined over the set of functions:

$$A = \{f(t)|\ \exists M, \tau_1, \tau_2 > 0, |f(t)| < Me^{\frac{|t|}{\tau_j}},\ if\ t \in (-1)^j \times [0,\infty)\}, \tag{2.1}$$

by the following formula

$$F(u) = S[f(t); u] = \int_0^\infty f(ut) e^{-t} dt, \quad u \in (-\tau, \tau). \tag{2.2}$$

The Sumudu transform the Caputo fractional derivative is defined as follows [7]:

$$S[D^\alpha f(t)] = u^{-\alpha} S[f(t)] - \sum_{k=0}^{m-1} u^{-\alpha+k} f^{(k)}(0^+), \quad m-1 < \alpha \leq m, \tag{2.3}$$

and for the Sumudu transform of the n-order derivative, we have

$$S\left[\frac{d^n f(t)}{dt^n}\right] = u^{-n}\left[S[f(t)] - \sum_{k=0}^{n-1} u^k \frac{d^k f(0)}{dt^k}\right]. \tag{2.4}$$

Some fundamental further established properties of Sumudu transform can be found in [1].

Theorem 2.1. Assuming $H(u) = S[h(t)]$ and $G(u) = S[g(t)]$, the Sumudu convolution theorem states that the transform of

$$h(t) * g(t) = \int_0^t h(t-\tau) g(\tau) d\tau, \tag{2.5}$$

is given by

$$uH(u)G(u) = S[h(t) * g(t)]. \tag{2.6}$$

Proof. see [3]. □

Definition 2.5. The Mittag-Leffler function $E_\alpha(z)$ with $\alpha > 0$ is defined by the following series representation, valid in the whole complex plane [25]:

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}.$$

3. Variational iteration method with Sumudu Transform

In order to elucidate the solution procedure of the variational iteration method, we consider the following general nonlinear fractional differential equation:

$$\frac{\partial^\alpha v(x,t)}{\partial t^\alpha} + \mathfrak{R}[x]v(x,t) + \mathcal{N}[x]v(x,t) = g(x,t) \quad t > 0,\ x \in \mathbb{R},\ 0 < \alpha \leq 1, \tag{3.1}$$
$$v(x,0) = h(x),$$

where $\mathfrak{R}[x]$ is the linear operator and $\mathcal{N}[x]$ is the general nonlinear operator. According to VIM introduced by He [13], the basic character of the method is to construct the following correction functional for (3.1):

$$v_{n+1}(x,t) = v_n(x,t) + \int_0^t \lambda(t,\tau) \left[\frac{d^\alpha v_n}{d\tau^\alpha} + \mathfrak{R}[\tilde{v}_n(x,\tau)] + \mathcal{N}[\tilde{v}_n(x,\tau)] - g(x,\tau) \right] d\tau, \tag{3.2}$$

where the function $\lambda(t,\tau)$ is called the Lagrange multiplier, which can be identified optimally via variational theory and v_n is the nth-order approximate solution. The initial values are usually used for selecting the zeroth approximation $v_0(x,t)$. With λ determined, several approximations v_j, $j > 0$ follows immediately. Consequently, the exact solution may be obtained by using

$$v(x,t) = \lim_{n \to \infty} v_n(x,t).$$

To solve (3.1) by variational iteration method, we first determine the Lagrange multiplier λ that will be identified optimally via integration by parts. But in fractional calculus, generally, the following integration by parts cannot hold:

$$J_t^\alpha v_0^C D^\alpha u = [uv]|_0^t - J_t^\alpha u_0^C D^\alpha v. \tag{3.3}$$

For this reason, we use of the Sumudu transform for finding the Lagrange multiplier.
Theorem 4.1. If the correction functional for (3.1) is established via the R-L integration

$$v_{n+1} = v_n + J_t^\alpha \lambda(t,\tau) \left[\frac{d^\alpha v_n}{d\tau^\alpha} + \mathfrak{R}[v_n(x,\tau)] + \mathcal{N}[v_n(x,\tau)] - g(x,\tau) \right], \tag{3.4}$$

the terms $\mathfrak{R}[v_n]$ and $\mathcal{N}[v_n]$ are restricted variations, the Lagrange multiplier can be identified as:

$$\lambda(t,\tau) = -1. \tag{3.5}$$

Proof. Take Sumudu transform on the both sides of (3.4)

$$V_{n+1}(x,u) = V_n(x,u) + S\left[J_t^{\alpha}\lambda(t,\tau)\left[\frac{d^{\alpha}v_n}{d\tau^{\alpha}} + \mathfrak{R}[v_n(x,\tau)] + \mathcal{N}[v_n(x,\tau)] - g(x,\tau)\right]\right]. \tag{3.6}$$

Now, consider the term

$$J_t^{\alpha}\lambda(t,\tau)\frac{d^{\alpha}v_n}{d\tau^{\alpha}} = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\lambda(t,\tau)\frac{d^{\alpha}v_n}{d\tau^{\alpha}}d\tau. \tag{3.7}$$

Setting the Lagrange multiplier $\lambda(t,\tau) = \lambda(X)|_{X=(t-\tau)}$, (3.7) is the convolution of the function $a(t) = \frac{\lambda(t)t^{\alpha-1}}{\Gamma(\alpha)}$ and the term $\frac{d^{\alpha}v_n}{d\tau^{\alpha}}$. The terms $\mathfrak{R}[v_n]$ and $\mathcal{N}[v_n]$ are considered as restricted variations which implies $\delta\mathfrak{R}[v_n] = 0$ and $\delta\mathcal{N}[v_n] = 0$, respectively.
Make the correction functional (3.6) stationary with respect to $V_n(x,u)$ and take the classical variation derivative δ on the both sides of (3.6). Then, we have

$$\delta V_{n+1}(x,u) = \delta V_n(x,u) + \delta\left[uA(u)u^{-\alpha}V_n(x,u) - uA(u)u^{-\alpha}v(x,0)\right]. \tag{3.8}$$

From (3.8), we can obtain the equation

$$1 + A(u)u^{-\alpha+1} = 0, \tag{3.9}$$

which results in

$$A(u) = \frac{-1}{u^{-\alpha+1}}. \tag{3.10}$$

Now, the Lagrange multiplier can be identified as:

$$\lambda(t,\tau) = \frac{a(t-\tau)\Gamma(\alpha)}{(t-\tau)^{\alpha-1}} = -1. \tag{3.11}$$

□

The above Lagrange multiplier was obtained by Laplace transform in [28].

4. New identification of the Lagrange multipliers

In this section, we use a distinct approach to the identification of the new Lagrange multipliers to fractional differential equations. Taking the above sumudu transform to both sides of (3.1); then the this equation is transformed into an algebraic equation as follows:

$$u^{-\alpha}V(x,u) - u^{-\alpha}v(x,0) + S\Big(\mathfrak{R}[v(x,t)] + \mathcal{N}[v(x,t)] - g(x,t)\Big) = 0, \tag{4.1}$$

where $V(x,u) = S[v(x,t)]$. With the original idea of the Lagrange multipliers, an iteration formula for (3.1) can be constructed as:

$$V_{n+1}(x,u) = V_n(x,u) + \lambda(u)\Big[u^{-\alpha}V_n(x,u) - u^{-\alpha}v(x,0) + S[\mathfrak{R}[v_n(x,t)]] + S[\mathcal{N}[v_n(x,t)]] - S[g(x,t)]\Big]. \tag{4.2}$$

Take the classical variation derivative δ on the both sides of (4.2). Considering $S(\mathfrak{R}[v_n])$ and $S(\mathcal{N}[v_n])$ as restricted terms, one can implies $\delta S(\mathfrak{R}[v_n]) = 0$ and $\delta S(\mathcal{N}[v_n]) = 0$, respectively. Therefore, we have

$$\delta V_{n+1}(x,u) = \delta V_n(x,u) + \delta\left[\lambda(u)u^{-\alpha}V_n(x,u)\right]. \tag{4.3}$$

The optimality condition for the extreme $\frac{\delta V_{n+1}}{\delta V_n} = 0$ from (4.3) leads to

$$\lambda(u) = -1/u^{-\alpha}. \tag{4.4}$$

By applying inverse-Sumudu transform, the iteration formula (4.2) can be explicitly given as:

$$\begin{aligned} v_{n+1}(x,t) =& v_n(x,t) - S^{-1}\left[\frac{1}{u^{-\alpha}}\left[u^{-\alpha}V_n(x,u) - u^{-\alpha}v(x,0)\right.\right.\\ &\left.\left. + S[\mathfrak{R}[v_n(x,t)]] + S[\mathcal{N}[v_n(x,t)]] - S[g(x,t)]\right]\right]\\ =& S^{-1}\left[v(x,0) - \left[\frac{1}{u^{-\alpha}}\left[S[\mathfrak{R}[v_n(x,t)]] + S[\mathcal{N}[v_n(x,t)]] - S[g(x,t)]\right]\right]\right], \end{aligned} \tag{4.5}$$

with the initial iteration $v_0(x,t) = v(x,0) = h(x)$.

5. Numerical examples

In this section, we discuss the implementation of our proposed algorithm and investigate its accuracy by applying the variational iteration method with coupling of the Sumudu transform. Some illustrative examples are presented to explain the efficiency and simplicity of the proposed method.

Example 5.1. Consider the following fractional Black-Scholes option pricing equation [19, 12]:

$$\frac{\partial^\alpha v}{\partial t^\alpha} = \frac{\partial^2 v}{\partial x^2} + (k-1)\frac{\partial v}{\partial x} - kv, \quad 0 < \alpha \leq 1, \tag{5.1}$$

with initial condition $v(x,0) = \max(e^x - 1, 0)$. Note that this system of equations contains just two dimensionless parameters $k = \frac{2r}{\sigma^2}$, where k represents the balance between the rate of interests and the variability of stock returns and the dimensionless time to expiry $\frac{\sigma^2 T}{2}$, even though there are four dimensional parameters, E, T, σ^2, and r, in the original statements of the problem.
Applying the Sumudu transform on both sides of (5.1), we get the following iteration formula:

$$\begin{aligned} V_{n+1}(x,u) = V_n(x,u) + \lambda(u)&\left[u^{-\alpha}V_n(x,u) - u^{-\alpha}v(x,0)\right.\\ &\left. + S\left(-\frac{\partial^2 v_n}{\partial x^2} - (k-1)\frac{\partial v_n}{\partial x} + kv_n\right)\right]. \end{aligned} \tag{5.2}$$

After the identification of a Lagrange multiplier $\lambda(u) = -1/u^{-\alpha}$, one can derive

$$v_{n+1}(x,t) = S^{-1}\left[v(x,0) - \left[\frac{1}{u^{-\alpha}} S\left(-\frac{\partial^2 v_n}{\partial x^2} - (k-1)\frac{\partial v_n}{\partial x} + kv_n\right)\right]\right]. \tag{5.3}$$

Now, we can obtain the following approximations:

$$\begin{aligned}
v_0(x,t) &= v(x,0) = \max(e^x - 1, 0),\\
v_1(x,t) &= \max(e^x - 1, 0) - S^{-1}\left[\frac{1}{u^{-\alpha}} S\left(-\frac{\partial^2 v_0}{\partial x^2} - (k-1)\frac{\partial v_0}{\partial x} + kv_0\right)\right]\\
&= \max(e^x - 1, 0) + \frac{t^\alpha}{\Gamma(1+\alpha)}[k\ \max(e^x, 0) - k\ \max(e^x - 1, 0)],\\
v_2(x,t) &= \max(e^x - 1, 0) - S^{-1}\left[\frac{1}{u^{-\alpha}} S\left(-\frac{\partial^2 v_0}{\partial x^2} - (k-1)\frac{\partial v_0}{\partial x} + kv_0\right)\right]\\
&= \max(e^x - 1, 0) + \frac{t^\alpha}{\Gamma(1+\alpha)}[k\ \max(e^x, 0) - k\ \max(e^x - 1, 0)]\\
&+ \frac{t^{2\alpha}}{\Gamma(1+2\alpha)}[-k^2\ \max(e^x, 0) + k^2\ \max(e^x - 1, 0)],\\
&\vdots\\
v_n(x,t) &= \max(e^x - 1, 0)\frac{t^\alpha}{\Gamma(1+\alpha)}[k\ \max(e^x, 0) - k\ \max(e^x - 1, 0)]\\
&+ \frac{t^{2\alpha}}{\Gamma(1+2\alpha)}[-k^2\ \max(e^x, 0) + k^2\ \max(e^x - 1, 0)] + ...\\
&+ \frac{t^{n\alpha}}{\Gamma(1+n\alpha)}[-(-k)^n\ \max(e^x, 0) + (-k)^n\ \max(e^x - 1, 0)],
\end{aligned} \tag{5.4}$$

the exact solution can be given in a compact form

$$v(x,t) = \lim_{n\to\infty} v_n(x,t) = \max(e^x - 1, 0)E_\alpha(-kt^\alpha) + \max(e^x, 0)(1 - E_\alpha(-kt^\alpha)), \tag{5.5}$$

where $E_\alpha(z)$ is Mittag-Leffler function in one parameter. The analytical solution of this problem is consistent with the result obtained by Kumar and et al. in [19]. For case $\alpha = 1$, we have

$$v(x,t) = \max(e^x - 1, 0)e^{-kt} + \max(e^x, 0)(1 - e^{-kt}), \tag{5.6}$$

which is an exact solution of the classic Black-Scholes equation.

Example 5.2. Consider the following generalized fractional Black-Scholes equation as follows [6]:

$$\frac{\partial^\alpha v}{\partial t^\alpha} + 0.08(2 + \sin(x))^2 x^2 \frac{\partial^2 v}{\partial x^2} + 0.06x\frac{\partial v}{\partial x} - 0.06v = 0, \tag{5.7}$$

with $0 < \alpha \leq 1$ and initial condition $v(x, 0) = \max(x - 25e^{-0.06}, 0)$.
In this example, we have

$$v_0(x, t) = \max(x - 25e^{-0.06}, 0). \tag{5.8}$$

Now, applying the Sumudu transform on both sides of (5.7), we have

$$\begin{aligned} V_{n+1}(x, u) =& V_n(x, u) + \lambda(u)\Big[u^{-\alpha}V_n(x, u) - u^{-\alpha}v(x, 0) \\ & + S\Big(\frac{\partial^\alpha v_n}{\partial t^\alpha} + 0.08(2 + \sin(x))^2 x^2 \frac{\partial^2 v_n}{\partial x^2} + 0.06x\frac{\partial v_n}{\partial x} - 0.06v_n\Big)\Big]. \end{aligned} \tag{5.9}$$

Operating the inverse Sumudu transform on both sides of (5.9) and considering $\lambda(u) = -1/u^{-\alpha}$, we can have the following iteration formula:

$$\begin{aligned} v_{n+1}(x, t) =& v_n(x, t) - S^{-1}\Big[\frac{1}{u^{-\alpha}}\Big[u^{-\alpha}V_n(x, u) - u^{-\alpha}v(x, 0) \\ & + S\left(\frac{\partial^\alpha v_n}{\partial t^\alpha} + 0.08(2 + \sin(x))^2 x^2 \frac{\partial^2 v_n}{\partial x^2} + 0.06x\frac{\partial v_n}{\partial x} - 0.06v_n\right)\Big]\Big] \\ =& S^{-1}\Big[v(x, 0) - \Big[\frac{1}{u^{-\alpha}}S\Big(\frac{\partial^\alpha v_n}{\partial t^\alpha} + 0.08(2 + \sin(x))^2 x^2 \frac{\partial^2 v_n}{\partial x^2} \\ & + 0.06x\frac{\partial v_n}{\partial x} - 0.06v_n\Big)\Big]\Big]. \end{aligned} \tag{5.10}$$

As a result, the successive approximation can be obtained as follows:

$$\begin{aligned} v_0(x, t) &= \max(x - 25e^{-0.06}, 0), \\ v_1(x, t) &= v_0(x, t) - S^{-1}\Big[\frac{1}{u^{-\alpha}}S\Big(\frac{\partial^\alpha v_0}{\partial t^\alpha} + 0.08(2 + \sin(x))^2 x^2 \frac{\partial^2 v_0}{\partial x^2} + 0.06x\frac{\partial v_0}{\partial x} - 0.06v_0\Big)\Big] \\ &= \max(x - 25e^{-0.06}, 0) + \frac{t^\alpha}{\Gamma(1 + \alpha)}[-0.06x + 0.06\ \max(x - 25e^{-0.06}, 0)], \\ v_2(x, t) &= v_0(x, t) - S^{-1}\Big[\frac{1}{u^{-\alpha}}S\Big(\frac{\partial^\alpha v_1}{\partial t^\alpha} + 0.08(2 + \sin(x))^2 x^2 \frac{\partial^2 v_1}{\partial x^2} + 0.06x\frac{\partial v_1}{\partial x} - 0.06v_1\Big)\Big] \\ &= \max(x - 25e^{-0.06}, 0) + \frac{t^\alpha}{\Gamma(1 + \alpha)}[-0.06x + 0.06\ \max(x - 25e^{-0.06}, 0)] \\ &+ \frac{t^{2\alpha}}{\Gamma(1 + 2\alpha)}[-(0.06)^2 x + (0.06)^2\ \max(x - 25e^{-0.06}, 0)], \end{aligned} \tag{5.11}$$

so that the solution $v(x, t)$ of the problem is given by:

$$v(x, t) = \lim_{n\to\infty} v_n(x, t) = \max(x - 25e^{-0.06}, 0)E_\alpha(0.06t^\alpha) + x(1 - E_\alpha(0.06t^\alpha)),$$

(5.12)

which is the exact solution of the given fractional Black-Scholes equation, for pricing the European option.
The exact solution of the given option pricing equation for $\alpha = 1$ is

$$v(x,t) = \max(x - 25e^{-0.06}, 0)e^{0.06t} + x(1 - e^{0.06t}). \tag{5.13}$$

6. Conclusion

In this paper, a concept of the Sumudu-Lagrange multipliers is successfully applied for pricing European option of the fractional Black-Scholes equation. This scheme was clearly very efficient and powerful technique in finding the solutions of the proposed equations. We note that the integration by parts is not used and the calculation of the Lagrange multiplier here is much simpler.

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